tkz-euclide 1.1

AlterMundus

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http://altermundus.com

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tkz-euclide

This document is a little gallery of some simple sangakus. I made all these pictures with my package **`tkz-euclide.sty`**. *Sangaku or San Gaku are colorful wooden tablets which were hung often in shinto shrines and sometimes in buddhist temples in Japan and posing typical and elegant mathematical problems. The problems featured on the sangaku are problems of japanese mathematics (wasan). The earliest sangaku found date back to the beginning of the 17th century.*

[Sangaku v1.5 2010/08/14]

☞ Firstly, I would like to thank **Till Tantau** for the beautiful LATEX package, namely TikZ.

☞ I am grateful to **Michel Bovani** for providing the **`fourier`** font.

☞ I received much valuable advice from **Jean-Côme Charpentier** and **Josselin Noirel**.

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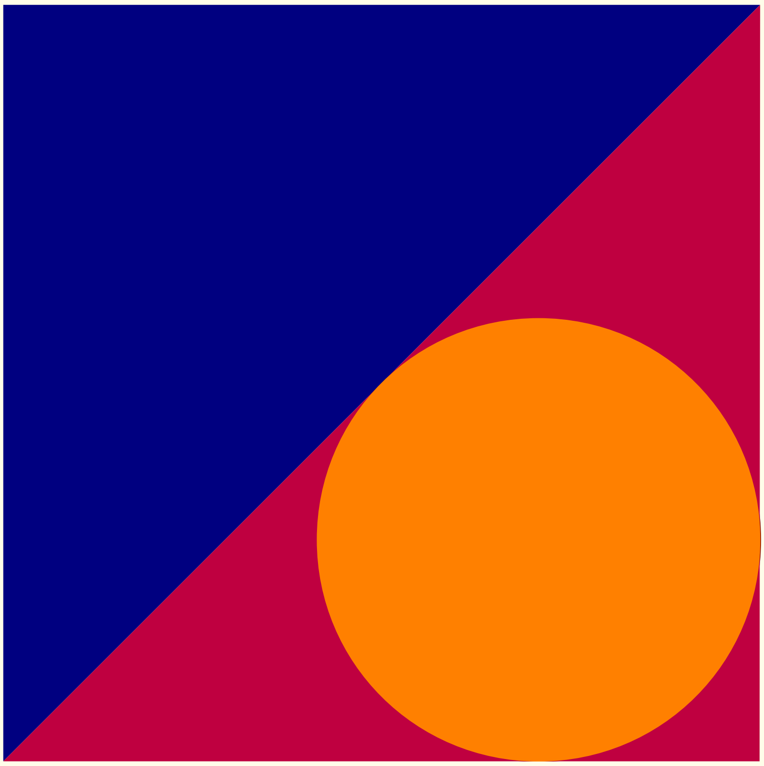
SECTION 1

Sangaku in a square - A very simple sangaku.

1.1 The picture

Find a relationship between the radius of the yellow circle and the side of the square.



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\begin{tikzpicture}[scale=1.25]
  \tkzInit \tkzClip \tkzDefPoint(0,0){A} \tkzDefPoint(8,0){B}
  \tkzDefSquare(A,B) \tkzGetPoints{C}{D}
  \tkzDefPointBy[projection=onto A--C](B) \tkzGetPoint{F}
  \tkzDefLine[bisector](A,C,B) \tkzGetPoint{c}
  \tkzInterLL(C,c)(B,F) \tkzGetPoint{I}  \tkzDrawCircle(I,F)
  \tkzCalcLength(I,F)\tkzGetLength{dIF}
  \tkzFillPolygon[color = blue!50!black](A,C,D)%
  \tkzFillPolygon[color = purple](A,B,C)%
  \tkzFillCircle[R,color = orange](I,\dIF pt)%
\end{tikzpicture}
```

1.2 Explanation

Firstly, we can found the relationship between the inradius and the sides of a right triangle.
If r is the inradius of a circle inscribed in a right triangle with sides a and b and hypotenuse c, then

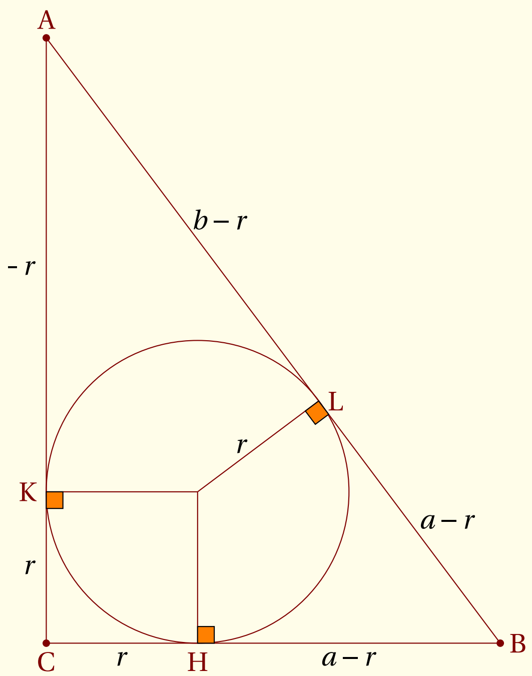
$$r = \frac{1}{2}(a+b-c).$$

Let ABC represents a right triangle, with the right angle located at C, as shown on the figure. Let a, b and c the lengths of the three sides; c is the length of the hypothenuse.
Let r and p be the radius of the incercle and the semiperimeter of the triangle.
a, b and c can be regarded in relation to r and they may be expressed with r: $a = r+(a-r)$, $b = r+(b-r)$ and $c = (a-r)+(b-r)$.
In a right triangle, we have the relation $r = p/2-c$. From the diagram, the hypotenuse AB is split in two pieces: $(a-r)$ and $(b-r)$, the length of the hypothenuse is $c = (a-r)+(b-r)$.
The perimeter is a function of r

$$p = a+b+c = r+(a-r)+r+(b-r)+(a-r)+(b-r) = 2a+2b-2r$$

so we can expressed r with s and c

$$2r = a+b-c = p-2c \text{ and } r = \frac{p}{2}-c = \frac{a+b-c}{2}$$

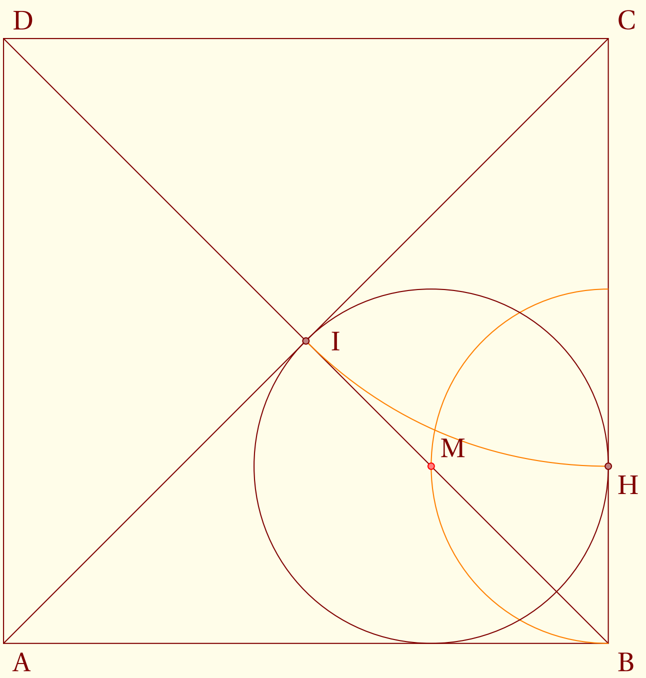



```
\begin{tikzpicture}
   \tkzInit\tkzClip[space = 0.5]
   \tkzPoint[pos = above](0,8){A}
   \tkzPoint[pos = right](6,0){B}
   \tkzPoint[pos = below](0,0){C}
   \tkzDrawPolygon(A,B,C)
   \tkzInCenter(A,B,C) \tkzGetPoint{I}
   \tkzDefPointBy[projection=onto B--C](I)
      \tkzGetPoint{H}
   \tkzDefPointBy[projection=onto A--C](I)
      \tkzGetPoint{K}
   \tkzDefPointBy[projection=onto A--B](I)
      \tkzGetPoint{L}
   \tkzDrawSegments(I,L I,H I,K)
   \tkzDrawCircle(I,H)
   \tkzMarkRightAngles%
   [fill=orange](I,L,B  B,H,I  C,K,I)
   \tkzLabelSegment[below](C,H){$r$}
   \tkzLabelSegment[below](H,B){$a-r$}
   \tkzLabelSegment[left](C,K){$r$}
   \tkzLabelSegment[left](K,A){$b-r$}
   \tkzLabelSegment[right](L,B){$a-r$}
   \tkzLabelSegment[right](A,L){$b-r$}
   \tkzLabelSegment[left](I,L){$r$}
   \tkzLabelPoint[below](H){$H$}
   \tkzLabelPoint[left](K){$K$}
   \tkzLabelPoint[right](L){$L$}
\end{tikzpicture}
```

Now, let ABC represents a isosceles right triangle with AB = AC = a, then AC = $\sqrt{2a}$ and $a+b-c = 2a-\sqrt{2}a$
So the inradius in this case is

$$r = \frac{2-\sqrt{2}}{2}a$$

Now we can obtain the incenter without the bisectors



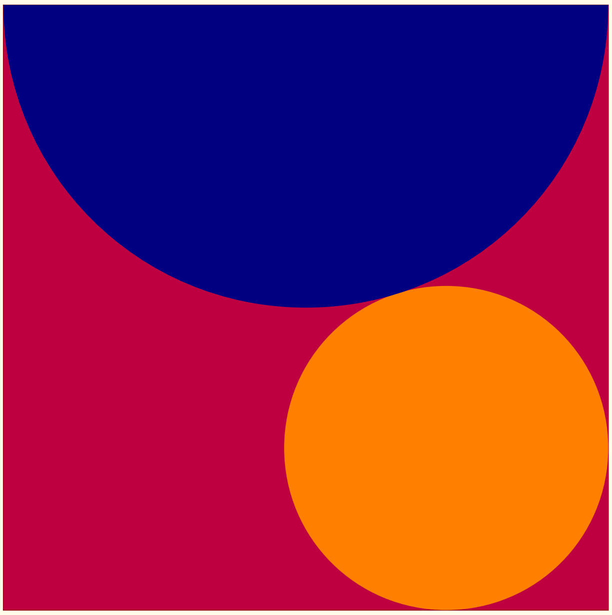

```
\begin{tikzpicture}
  \tkzInit[xmin = -1,ymin = -1,xmax = 9,ymax = 9]\tkzClip
  \tkzDefPoint(0,0){A} \tkzDefPoint(8,0){B}
  \tkzDefSquare(A,B)\tkzGetPoints{C}{D}
  \tkzDefMidPoint(A,C) \tkzGetPoint{I}
  \tkzDuplicateSegment(C,I)(C,B)\tkzGetPoint{H}
  \tkzInterLC(B,D)(H,B) \tkzGetPoints{M}{N}
  \tkzDrawPolygon(A,B,C,D)
  \tkzDrawSegments(A,C B,D)
  \tkzDrawArc(C,I)(H)
  \tkzDrawArc[rotate](H,B)(-180)
  \tkzDrawCircle(M,H)
  \tkzDrawPoints(I,H)
  \tkzDrawPoint[color = red](M)
  \tkzLabelPoints(A,B,H)
  \tkzLabelPoints[right=6pt](I)
  \tkzLabelPoints[above right](C,D,M)
\end{tikzpicture}
```

SECTION 2

Sangaku in a square - Circle and semicircle

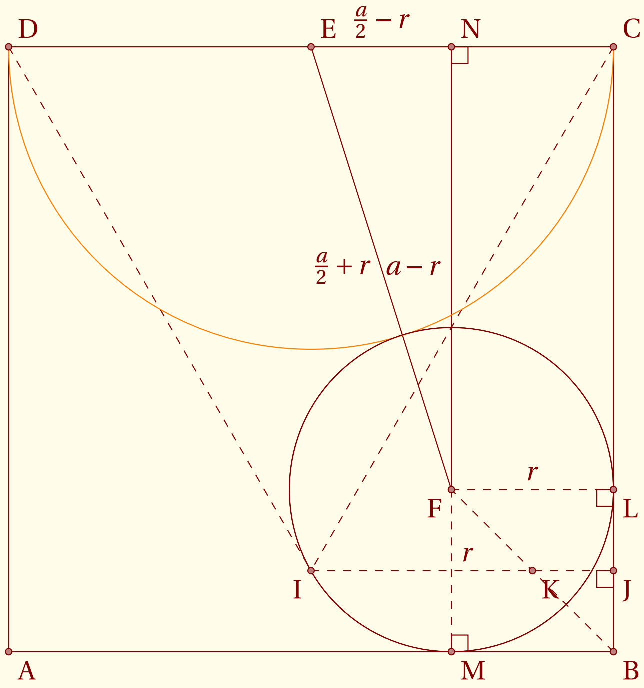
Find a relationship between the radius of the yellow circle and the side of the square.

2.1 The picture



```
\begin{tikzpicture}
   \tkzInit
   \tkzDefPoint(0,0){A} \tkzDefPoint(8,0){B}
   \tkzDefSquare(A,B) \tkzGetPoints{C}{D}
   \tkzDrawPolygon(B,C,D,A) \tkzClipPolygon(A,B,C,D)
   \tkzDefPoint(4,8){F}
   \tkzDefTriangle[equilateral](C,D) \tkzGetPoint{I}
   \tkzDefPointBy[projection=onto C--B](I) \tkzGetPoint{J}
   \tkzInterLL(D,B)(I,J) \tkzGetPoint{K}
     \tkzDefPointBy[symmetry=center K](B)
   \tkzGetPoint{M}
   \tkzCalcLength(M,I)\tkzGetLength{dMI}   \tkzDrawPoint(I)
   \tkzDrawCircle(M,I)
   \tkzFillPolygon[color = purple](A,B,C,D)
   \tkzFillCircle[R,color = orange](M,\dMI pt)
   \tkzFillCircle[R,color = blue!50!black](F,4 cm)%
\end{tikzpicture}
```

2.2 Explanation



FNE is a right triangle with hypotenuse [EF]. We have, $EN^2 + NF^2 = EF^2$ by the Pythagorean theorem. In terms of a and r, the theorem appears as

$$\left(\frac{a}{2} - r\right)^2 + (a - r)^2 = \left(\frac{a}{2} + r\right)^2$$

which is equivalent to

$$r^2 - 4ar + a^2 = r^2 - 4ar + 4a^2 - 3a^2 = (r - 2a - \sqrt{3}a)(r - 2a + \sqrt{3}a) = 0$$

And finally

$$r = a(2 - \sqrt{3}) < a$$

2.3 Construction

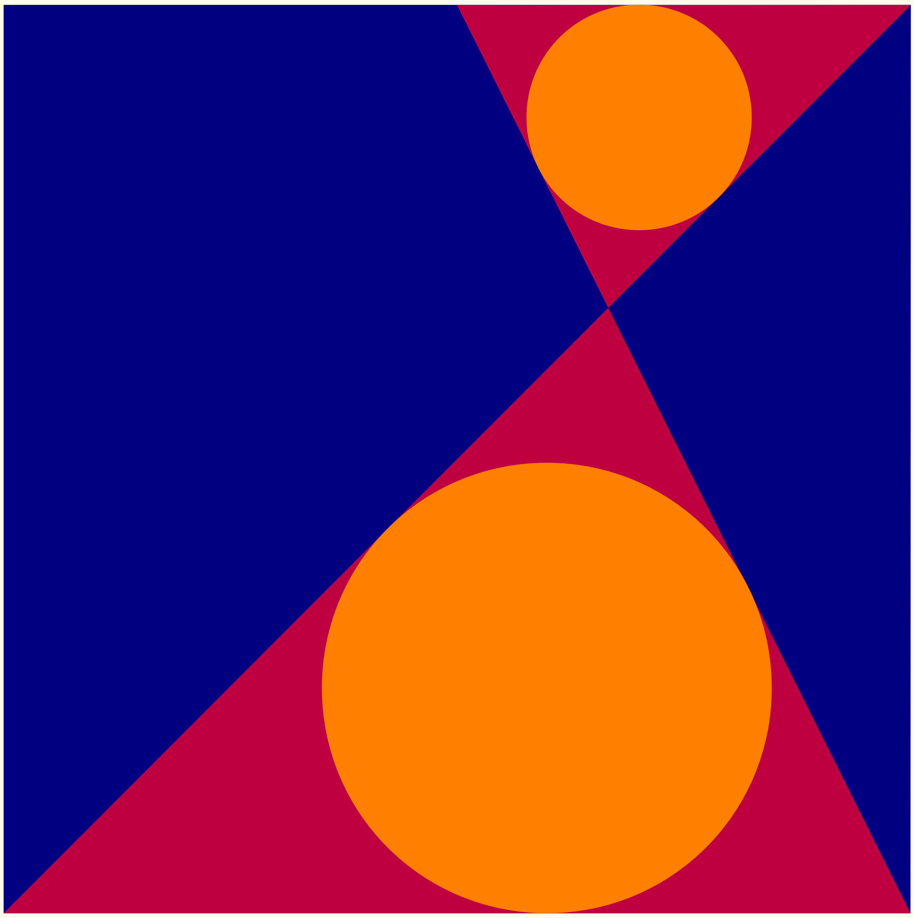
It's easy to prove : $BJ = a(1 - \frac{\sqrt{3}}{2})$ and $BL = a(2 - \sqrt{3})$. First we need to find the point I, then J. K is the intersection of (IJ) and the bisector line of the angle $\widehat{ABC}$. Finally, F is the symmetric point of B around K.

SECTION 3

Sangaku in a square - two inscribed circles

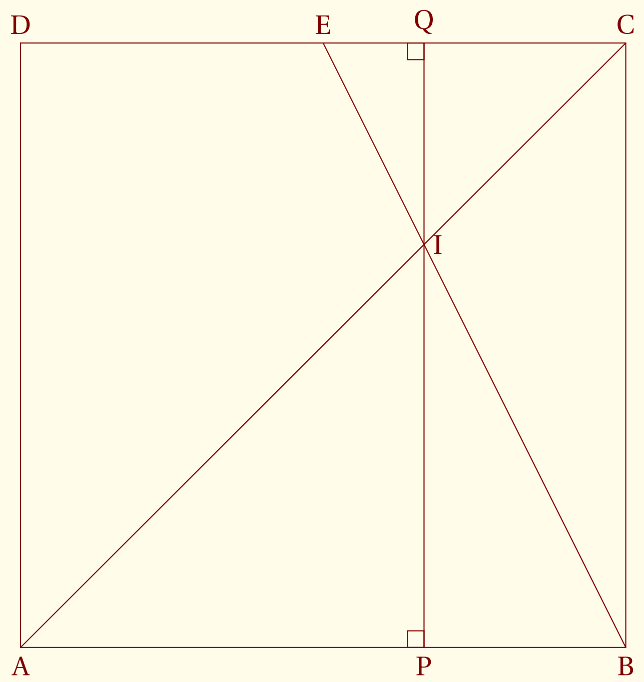
In the following diagram, a triangle is formed by a line that joins the base of a square with the midpoint of the opposite side and a diagonal. Find the radius of the two inscribed circles.

3.1 The picture



```
\begin{tikzpicture}[scale = 1.5]
   \tkzInit
   \tkzDefPoint(0,0){A}
   \tkzDefPoint(8,0){B}
   \tkzDefPoint(4,8){E}
   \tkzDefSquare(A,B) \tkzGetPoints{C}{D}
   \tkzDefLine[bisector](B,A,C) \tkzGetPoint{a}
   \tkzDefLine[bisector](E,B,A) \tkzGetPoint{b}
   \tkzInterLL(A,a)(B,b)  \tkzGetPoint{K}
   \tkzDefPointBy[projection=onto A--B](K) \tkzGetPoint{H}
   \tkzDefLine[bisector](D,C,A) \tkzGetPoint{c}
   \tkzDefLine[bisector](B,E,C) \tkzGetPoint{e}
   \tkzInterLL(C,c)(E,e)  \tkzGetPoint{K1}
   \tkzDefPointBy[projection=onto C--D](K1) \tkzGetPoint{H1}
   \tkzFillPolygon[color = blue!50!black](A,B,C,D)
   \tkzFillPolygon[color = purple](A,C,E,B)
   \tkzFillCircle[color = orange](K,H)
   \tkzFillCircle[color = orange](K1,H1)
\end{tikzpicture}
```

3.2 Explanation and construction



QC is parallel to the base AB and is half as long which implies that the two triangles QIC and QAB are similar. I divides the segments QP and AD in ratio 2:1 so that

$$\mathrm{IP} = \frac{2}{3}\mathrm{QP} = \frac{2a}{3}$$

$$\mathrm{AI} = \frac{2}{3}\mathrm{AC}$$

Thus assuming $\mathrm{AB} = a$, we have $\mathrm{AC} = \sqrt{2}a$, $\mathrm{AI} = \frac{2\sqrt{2}}{3}a$ and $\mathrm{BI} = \frac{2}{3}\mathrm{BE}$.
We can apply the Pythagorean theorem to find BE

$$\mathrm{BE} = \frac{\sqrt{5}}{2}a \quad \text{this means that} \quad \mathrm{BI} = \frac{\sqrt{5}}{3}a$$

In any triangle, $r \times p = s \times h$, where r is the inradius, p the perimeter, s the side and h the altitude of the triangle.
In other words

$$r\left(1 + \frac{\sqrt{5}}{3} + \frac{2\sqrt{2}}{3}\right)a = a \times \frac{2a}{3}$$

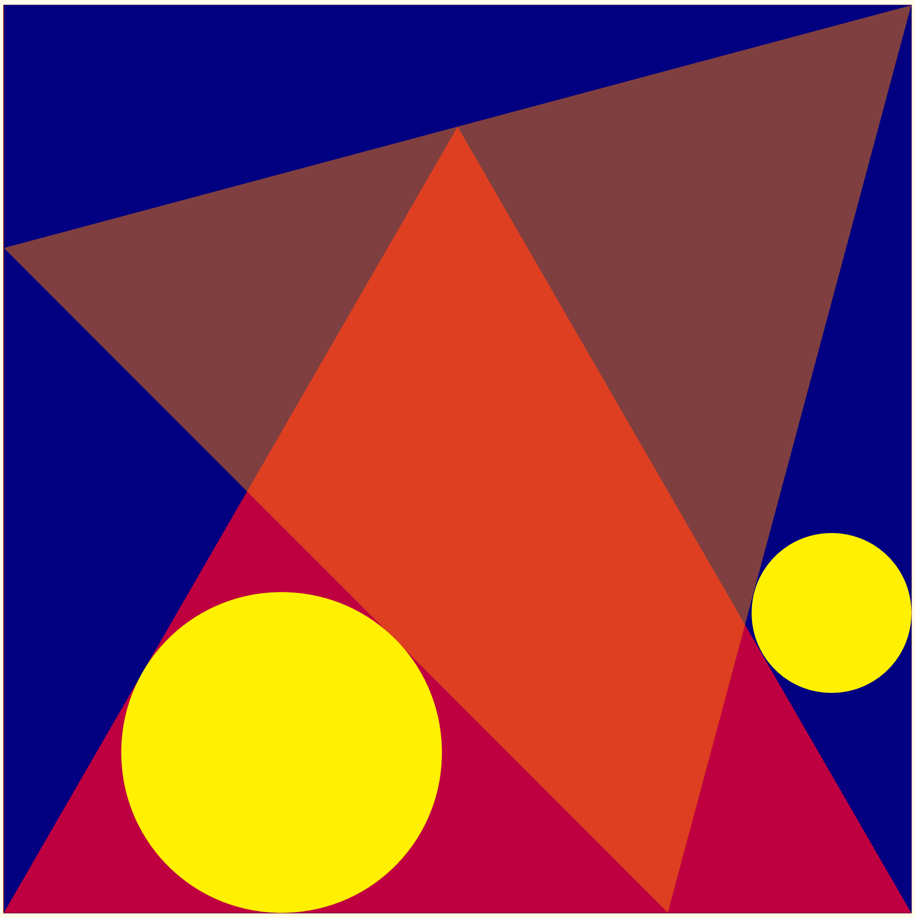
from which r is found:

$$r = \frac{2a}{3 + 2\sqrt{2} + \sqrt{5}}$$

SECTION 4

Sangaku in a square - Two equilateral triangles

Here is an elegant sangaku that requires both geometric and algebraic skills and some perseverance: Two equilateral triangles are inscribed into a square as shown in the diagram. Their side lines cut the square into a quadrilateral and a few triangles. Find a relationship between the radii of the two incircles shown in the diagram.



The code of the last figure is

```
\begin{tikzpicture}[scale = 1.5]
   \tkzClip[space=1]
   \tkzDefPoint(0,0){B} \tkzDefPoint(8,0){C}%
   \tkzDefPoint(0,8){A} \tkzDefPoint(8,8){D}
   \tkzDrawPolygon(B,C,D,A)
   \tkzClipPolygon(A,B,C,D)
   \tkzFillPolygon[color = blue!50!black](A,B,C,D)
   \tkzDefTriangle[equilateral](B,C) \tkzGetPoint{M}
   \tkzFillPolygon[color = purple](B,C,M)
   \tkzInterLL(D,M)(A,B)          \tkzGetPoint{N}
   \tkzDefPointBy[rotation=center N angle -60](D) \tkzGetPoint{L}
   \tkzDefLine[bisector](C,B,M)   \tkzGetPoint{x}
   \tkzDefLine[bisector](N,L,B)   \tkzGetPoint{y}
   \tkzInterLL(L,y)(B,x)          \tkzGetPoint{H}
   \tkzDefLine[bisector](M,C,D)   \tkzGetPoint{u}
   \tkzDefLine[bisector](L,D,C)   \tkzGetPoint{v}
   \tkzDefPointBy[projection=onto C--B](H) \tkzGetPoint{I}
   \tkzFillPolygon[color = orange,opacity = .5](D,N,L)
   \tkzFillCircle[color = yellow](H,I)
   \tkzInterLL(C,u)(D,v)  \tkzGetPoint{K}
   \tkzDefPointBy[projection=onto C--D](K) \tkzGetPoint{J}
   \tkzFillCircle[color = yellow](K,J)
\end{tikzpicture}
```

Firstly, we need to prove that it is possible that two equilateral triangles are inscribed into a square as shown in the diagram. A theorem exists but it is nice to find a solution in this particuler case. Let ABCD a square, BCM an equilateral triangle. The line (DM) intersects [AB] at point N. Then we construct a point L on the side [BC] and the angle $\widehat{NDL} = 60°$

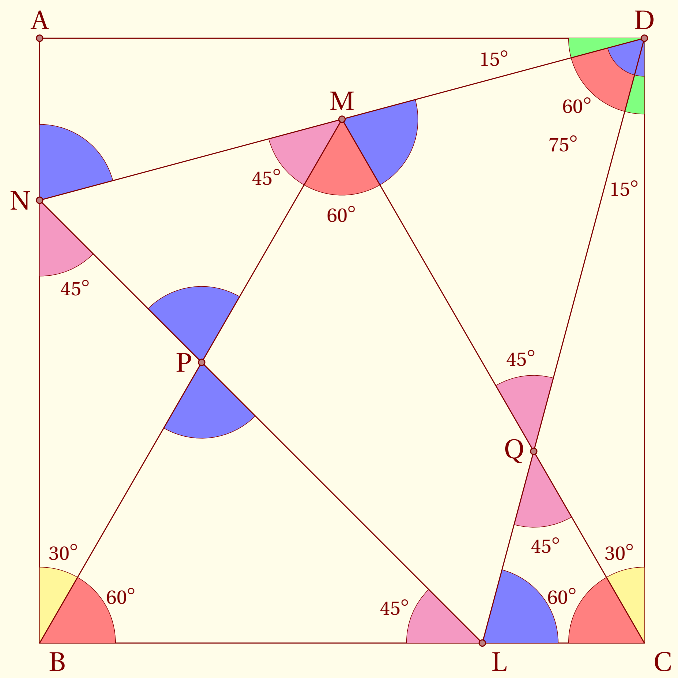
The triangle MCD is an isosceles triangle with two sides MC and CD of the same length a. It follows that

$\widehat{MDC} = \widehat{DMC} = 75°$ because $\widehat{MCD} = 30°$

Now we we can determine the angular size of all the angles

$\widehat{LDC} = 15°$ so $\widehat{ADN} = 15°$ and $\widehat{AND} = 75°$

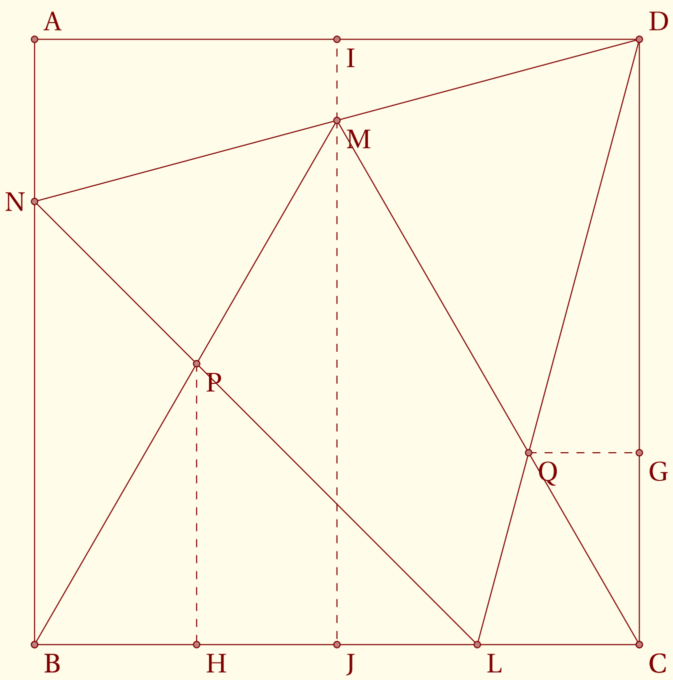
AN = LC then BL = BN and $\widehat{BLN} = \widehat{LNB} = \widehat{NMB} = \widehat{MQD} = \widehat{LQC} = 45°$




```
\begin{tikzpicture}
   \tkzInit
   \tkzDefPoint(0,0){B} \tkzDefPoint(8,0){C}
   \tkzDefPoint(0,8){A} \tkzDefPoint(8,8){D}
   \tkzDrawPolygon(B,C,D,A)
   \tkzDefTriangle[equilateral](B,C) \tkzGetPoint{M}
   \tkzInterLL(D,M)(A,B) \tkzGetPoint{N}
   \tkzDefPointBy[rotation=center N angle -60](D) \tkzGetPoint{L}
   \tkzInterLL(N,L)(M,B)     \tkzGetPoint{P}
   \tkzInterLL(M,C)(D,L)     \tkzGetPoint{Q}
   \tkzMarkAngles[fill = green!50](L,D,C A,D,N)
   \tkzMarkAngles[fill = yellow!50](D,C,M M,B,A)
   \tkzMarkAngles[fill = magenta!50](N,L,B B,N,L N,M,B D,Q,M L,Q,C)
   \tkzMarkAngles[fill = blue!50](D,N,A M,P,N B,P,L C,L,D C,M,D)
   \tkzMarkAngles[fill = red!50](C,B,M B,M,C M,C,B N,D,L)
   \tkzMarkAngle[size = 0.5,fill = blue!50](M,D,C)
   \tkzLabelAngle[dist=2](L,D,C){\scriptsize $15^{\circ}$}
   \tkzLabelAngle[dist=-2](A,D,N){\scriptsize $15^{\circ}$}
   \tkzLabelAngles[dist=1.25](D,C,M M,B,A){\scriptsize $30^{\circ}$}
   \tkzLabelAngles[dist=1.25](N,L,B B,N,L N,M,B D,Q,M L,Q,C){\scriptsize $45^{\circ}$}
   \tkzLabelAngles[dist=1.25](C,B,M B,M,C M,C,B N,D,L){\scriptsize $60^{\circ}$}
   \tkzLabelAngle[dist=1.75](M,D,C){\scriptsize $75^{\circ}$}
   \tkzDrawSegments(D,N N,L L,D B,M M,C)
   \tkzDrawPoints(L,N,P,Q,M,A,D) \tkzLabelPoints[left](N,P,Q)
   \tkzLabelPoints[above](M,A,D)  \tkzLabelPoints(L,B,C)
\end{tikzpicture}
```

4.1 Explanation

We can see that the angles $\widehat{\mathrm{DNL}}$ and $\widehat{\mathrm{NLD}}$ have the same degree measurements 60°. DNL is an equilateral triangle and it is the largest equilateral triangle which can be inscribed in the square (Madachy 1979). We prove lately that the side is $s = (\sqrt{6} - \sqrt{2})a$.



We need some preliminaries to find the ratio between the radii of the two incircles shown in the first diagram.
Assume the side of the square equals a, first, we determine MI

$$\mathrm{MJ} = \frac{\sqrt{3}}{2}a \text{ and } \mathrm{MI} = a - \frac{\sqrt{3}}{2}a = \frac{(2-\sqrt{3})}{2}a$$

Thus we can find AN and NB

$$\mathrm{AN} = 2\mathrm{MI} = (2-\sqrt{3})a$$

and

$$\mathrm{BN} = \mathrm{AB} - \mathrm{AN} = a - (2-\sqrt{3})a = (\sqrt{3}-1)a$$

ADN is a right triangle with hypotenuse ND. We have, $\mathrm{AD}^2 + \mathrm{AN}^2 = \mathrm{ND}^2$ by the Pythagorean theorem.

Using this, we continue:

$$\mathrm{ND}^2 = a^2 + (2-\sqrt{3})^2 a^2 = a^2(8-4\sqrt{3})$$

$$\mathrm{ND} = \mathrm{NL} = \mathrm{LD} = (\sqrt{6}-\sqrt{2})a$$

The value of $\tan(15^\circ)$ which will be useful later on.

$$\tan 15^\circ = \frac{\mathrm{AN}}{\mathrm{AD}} = \frac{(2-\sqrt{3})a}{a} = 2-\sqrt{3}$$

Now we can apply the standard formula in a triangle to determine the inradius

$$r = \frac{sh}{p}$$

where r, p, s, h are respectively the inradius, perimeter, a side and the altitude to the side in a triangle.
A good idea is to find a relationship between p and h
Let BPL the first triangle, here $h = \mathrm{PH}$, $l = \mathrm{BL}$ and $p = \mathrm{BP} + \mathrm{PL} + \mathrm{LB}$.

$$\sin(60^\circ) = \frac{\sqrt{3}}{2} = \frac{\mathrm{PH}}{\mathrm{BP}} = \frac{h}{\mathrm{BP}}$$

thus

$$\mathrm{BP} = \frac{2h}{\sqrt{3}}$$

HPL is an isosceles right triangle. The hypotenuse PL has length $\sqrt{2}h$.
And finally the relation between BL and h can be obtain like this

$$\mathrm{BL} = \mathrm{BH} + \mathrm{HL} = \frac{h}{\sqrt{3}} + h$$

In an other way

$$\mathrm{BL} = \mathrm{BN} = (\sqrt{3}-1)a$$

The inradius r_1 of BLP is

$$r_1 = \frac{sh}{p} = \frac{(\sqrt{3}-1)ah}{\frac{2h}{\sqrt{3}} + \sqrt{2}h + \frac{h}{\sqrt{3}} + h} = \frac{(\sqrt{3}-1)a}{1+\sqrt{2}+\sqrt{3}}$$

For CQD, similarly the inradius r_2 can be found with

$$r_2 = \frac{ah}{p}$$

with $p = \mathrm{CQ} + \mathrm{QD} + \mathrm{DC}$ and $h = \mathrm{QG}$
or

$$\frac{\mathrm{QD}}{\mathrm{DL}} = \frac{\mathrm{QG}}{\mathrm{LC}} = \frac{h}{(2-\sqrt{3})a}$$

from which

$$\mathrm{QD} = \frac{(\sqrt{6}-\sqrt{2})ah}{(2-\sqrt{3})a} = (\sqrt{6}+\sqrt{2})h$$

We continue

$$\frac{\mathrm{QG}}{\mathrm{QC}} = \frac{1}{2} = \frac{h}{\mathrm{GC}}$$

from which

$$\mathrm{QC} = 2h$$

and finally $\mathrm{DC} = \mathrm{DG} + \mathrm{GC} =$

$$\frac{h}{\mathrm{DG}} = \tan(15^\circ) = 2-\sqrt{3}$$

from which

$$\mathrm{DG} = \frac{h}{2-\sqrt{3}} = (2+\sqrt{3})h$$

and

$$\frac{\mathrm{GC}}{\mathrm{QC}} = \frac{\mathrm{GC}}{2h} = \frac{\sqrt{3}}{2}$$

Thus we can find

$$\mathrm{GC} = \sqrt{3}h$$

Finally

$$p = (\sqrt{6}+\sqrt{2})h + 2h + (2+\sqrt{3})h + \sqrt{3}h$$

The second radius is

$$r_2 = \frac{ah}{p} = \frac{ah}{(\sqrt{6}+\sqrt{2})h + 2h + (2+\sqrt{3})h + \sqrt{3}h} = \frac{a}{4+\sqrt{2}+2\sqrt{3}+\sqrt{6}}$$

Without a special effort, we conclude that $r_1 = 2r_2$

4.2 Construction

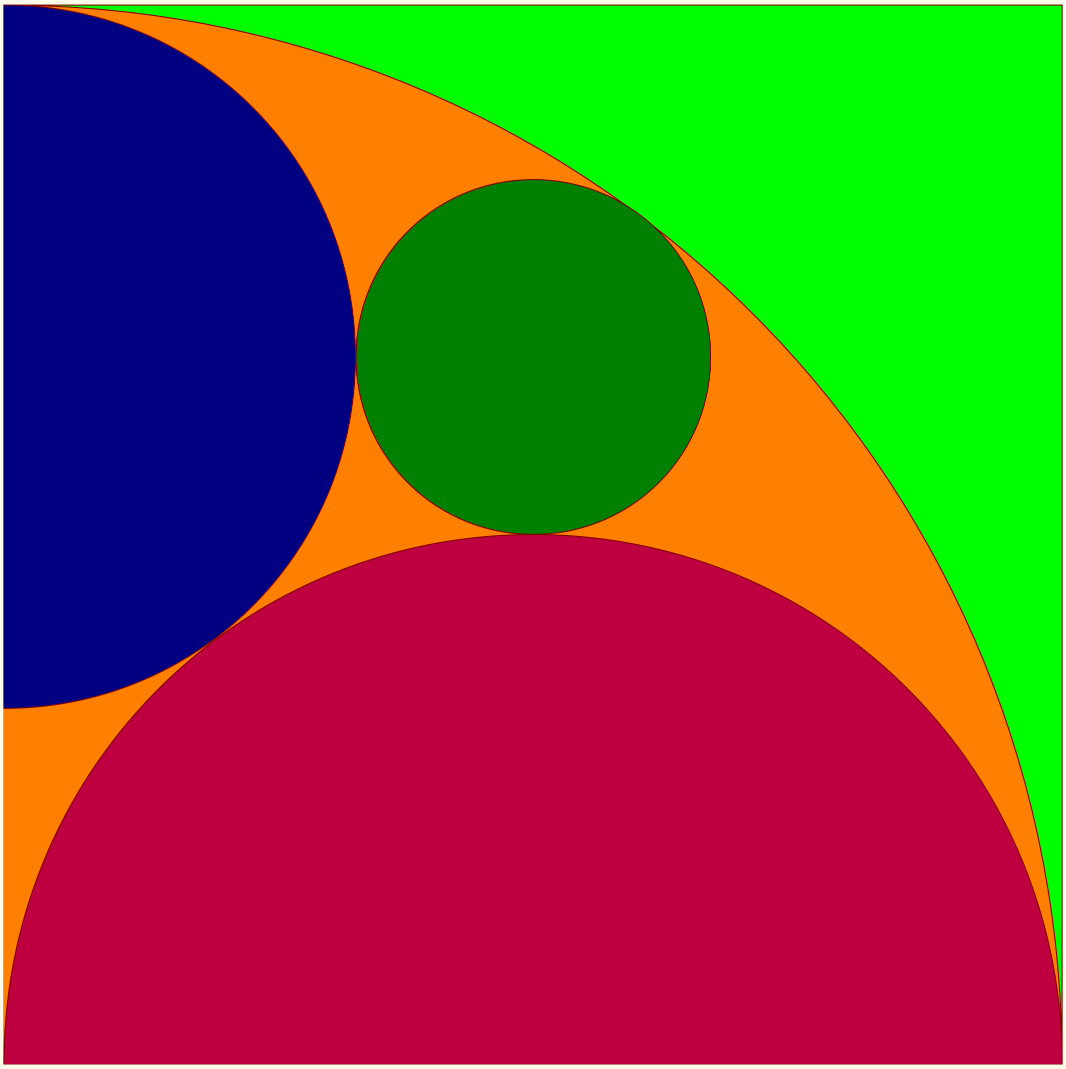
There is no problem. First we draw the square ABCD and an equilateral triangle BMC, then we draw the line (CM). N is the intersection of (CM) with (AB). We draw the equilateral triangle CNL. P et Q are intersections of sides of the equilateral triangles. Finally we draw the incenters of the triangles BPL and CQD.

SECTION 5

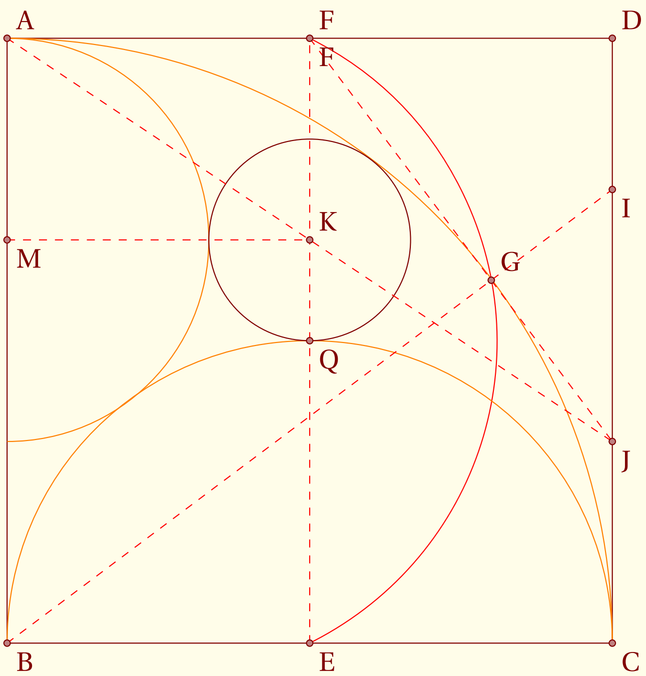
Sangaku in a square - V

This sangaku requires to determine the relative radii of the circles shown can be solved by an application of the Pythagorean theorem. Find a relationship between the radius of the circle and the side of the square.

5.1 The picture



```
\begin{tikzpicture}[scale = 1.75]
   \tkzDefPoint(0,0){B} \tkzDefPoint(8,0){C}%
   \tkzDefPoint(0,8){A} \tkzDefPoint(8,8){D}
   \tkzDrawPolygon[fill=green](A,B,C,D)
   \tkzClipPolygon(A,B,C,D)
   \tkzDefPoint(4,8){F}
   \tkzDefPoint(4,0){E}
   \tkzDefPoint(4,4){Q}
   \tkzTangent[from=B](F,A)  \tkzGetPoints{G}{H}
   \tkzInterLL(F,G)(C,D)   \tkzGetPoint{J}
   \tkzInterLL(A,J)(F,E)   \tkzGetPoint{K}
   \tkzDefPointBy[projection=onto B--A](K) \tkzGetPoint{M}
   \tkzDrawCircle[fill = orange](B,A)
   \tkzDrawCircle[fill = blue!50!black](M,A)
   \tkzDrawCircle[fill = purple](E,B)
   \tkzDrawCircle[fill = yellow](K,Q)
   \end{tikzpicture}
```



5.2 Explanation and construction

Assume the radius AM equal r and the side of the square is a.

step 1. Firstly, $\mathrm{ME}^2 = \mathrm{BE}^2 + \mathrm{BM}^2$ and the two circles are tangent if $\mathrm{ME} = r + \frac{a}{2}$. The equality becomes

$$\left(r + \frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2 + (a - r)^2$$

$$r = \frac{a}{3}$$

we have

$$\mathrm{ME} = \mathrm{BK} = \frac{5}{6}a$$

step 2.

$$\mathrm{KQ} = \mathrm{KE} - \mathrm{QE} = \frac{2a}{3} - \frac{a}{2} = \frac{a}{6}$$

and

$$\mathrm{MK} - \frac{a}{3} = \frac{a}{2} - \frac{a}{3} = \frac{a}{6}$$

The circle K is tangent mutually at the circle E and the circle M.

step 3. The little circle with center K is tangent at the circle B.

$$a - \frac{5a}{6} = \frac{a}{6}$$

step 4. A good method to obtain K is finally to place I such as $\mathrm{DI} = \frac{a}{4}$. Therefore, BI intercepts the big circle in G with $\mathrm{GI} = \frac{a}{4}$,$\mathrm{FG} = \frac{a}{2}$ and FG orthogonal to BG. FG intercepts BC in J such as $\mathrm{DJ} = \frac{2a}{3}$. K is the common point between AJ and EF.

SECTION 6

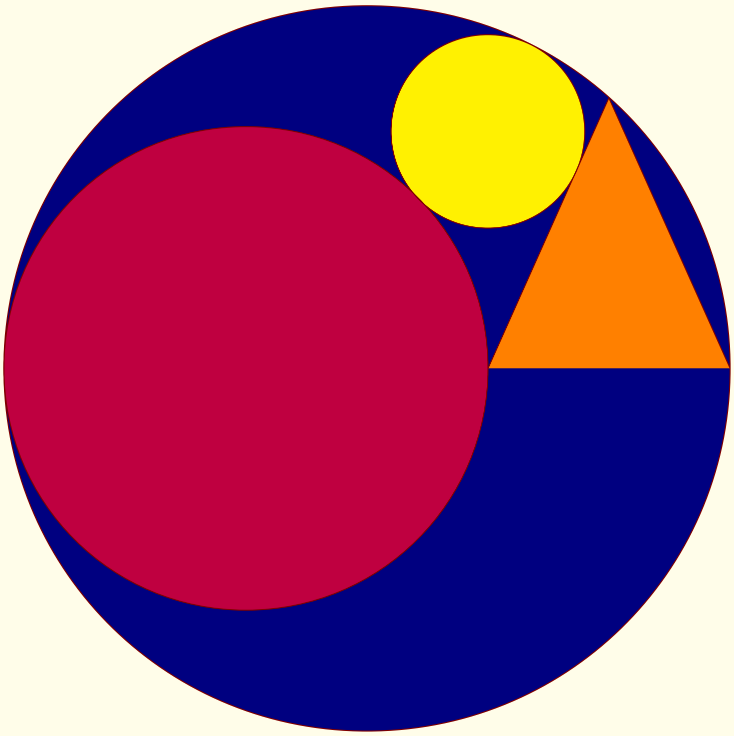
Sangaku in a square - Circle Inscribing

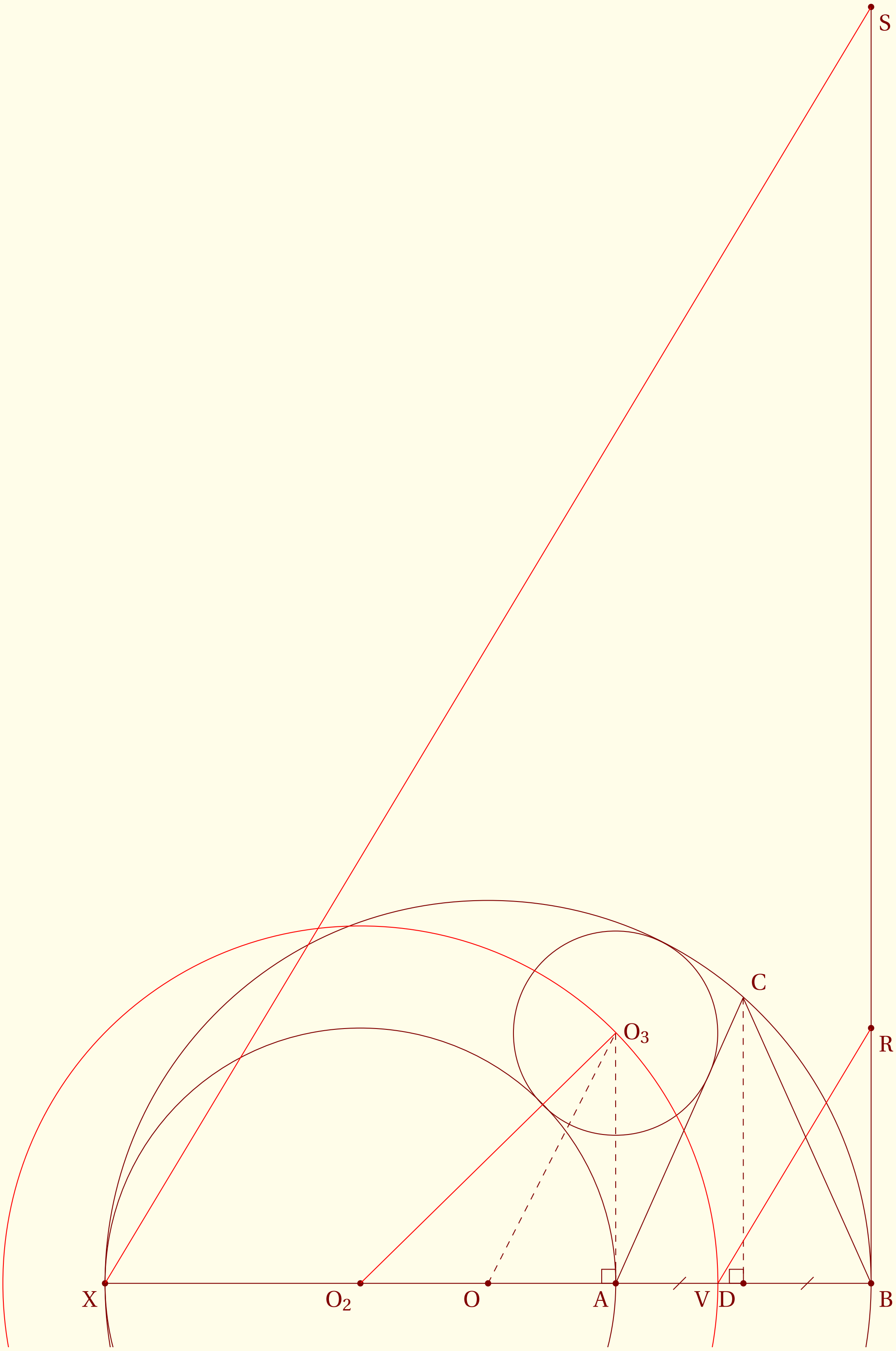
6.1 Sangaku - Circle Inscribing

Construct the figure consisting of a circle centered at O, *a second smaller circle centered at* O_2 *tangent to the first, and an isosceles triangle whose base* [AB] *completes the diameter of the larger circle* [XB] *through the smaller* [XA]. *Now inscribe a third circle with center* O_3 *inside the large circle, outside the small one, and on the side of a leg of the triangle.*

References
Weisstein, Eric W. "Circle Inscribing." From MathWorld–A Wolfram Web
http://mathworld.wolfram.com/CircleInscribing.html
Alexander Bogomolny
http://www.cut-the-knot.org/

In this problem, from an 1803 sangaku found in Gumma Prefecture, the base of an isosceles triangle sits on a diameter of the large blue circle. This diameter also bisects the purple circle, which is inscribed so that it just touches the inside of the blue circle and one vertex of the orange triangle, as shown. The yellow circle is inscribed so that it touches the outsides of both the purple circle and the triangle, as well as the inside of the blue circle. A line segment connects the center of the yellow circle and the intersection point between the purple circle and the orange triangle. Show that this line segment is perpendicular to the drawn diameter of the blue circle.




S
C
R
O3
X
O2
O
A
V
D
B

To find the explicit position and size of the circle, let the circle with center O have radius R and be centered at O and let the circle with center O_2 have radius r.

$$(r+a)^2 = r^2 + y^2$$

$$(R-a)^2 = (R-2r)^2 + y^2$$

for a and y gives

$$a = 2r\frac{R-r}{R+r}$$

$$y = AO_3 = 2\sqrt{2Rr}\frac{\sqrt{R-r}}{R+r}$$

but now we need to prove that the circle is tangent to the line AC.
Let α the angle ACD and the angle O_3AC

$$\sin(\alpha) = \frac{AD}{AC}$$

$OD = r$ and $AD = R - r$
OCD is a right triangle with hypotenuse $OC = R$. We have, $OD^2 + CD^2 = OC^2$ by the Pythagorean theorem. In terms of r, the theorem appears as

$$r^2 + CD^2 = R^2$$

which is equivalent to

$$CD^2 = R^2 - r^2$$

and with the right triangle ADC and the Pythagorean theorem

$$AC^2 = AD^2 + CD^2 = (R-r)^2 + R^2 - r^2 = 2R(R-r)$$

finally

$$\sin(\alpha) = \frac{AD}{AC} = \frac{R-r}{\sqrt{2R(R-r)}} = \frac{\sqrt{R-r}}{\sqrt{2R}}$$

Let H the projection point of O_3 on the line AC, and d the length of O_3H

$$\sin(\alpha) = \frac{O_3H}{AO_3} = \frac{d}{y} = \frac{d}{2\sqrt{2Rr}\frac{\sqrt{R-r}}{R+r}}$$

Using the two forms of sin(α)

$$\frac{d}{2\sqrt{2Rr}\frac{\sqrt{R-r}}{R+r}} = \frac{\sqrt{R-r}}{\sqrt{2R}}$$

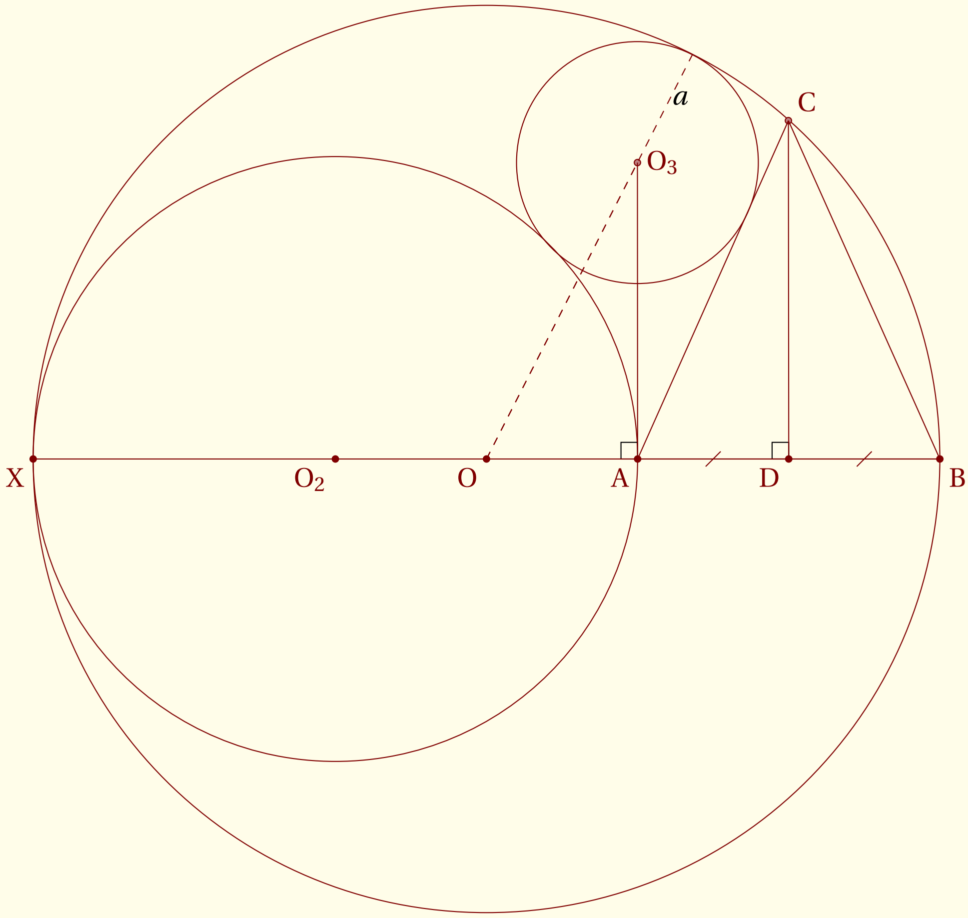
So

$$d = 2r\frac{R-r}{R+r} = a$$

The code on the page is needed to get the examples.

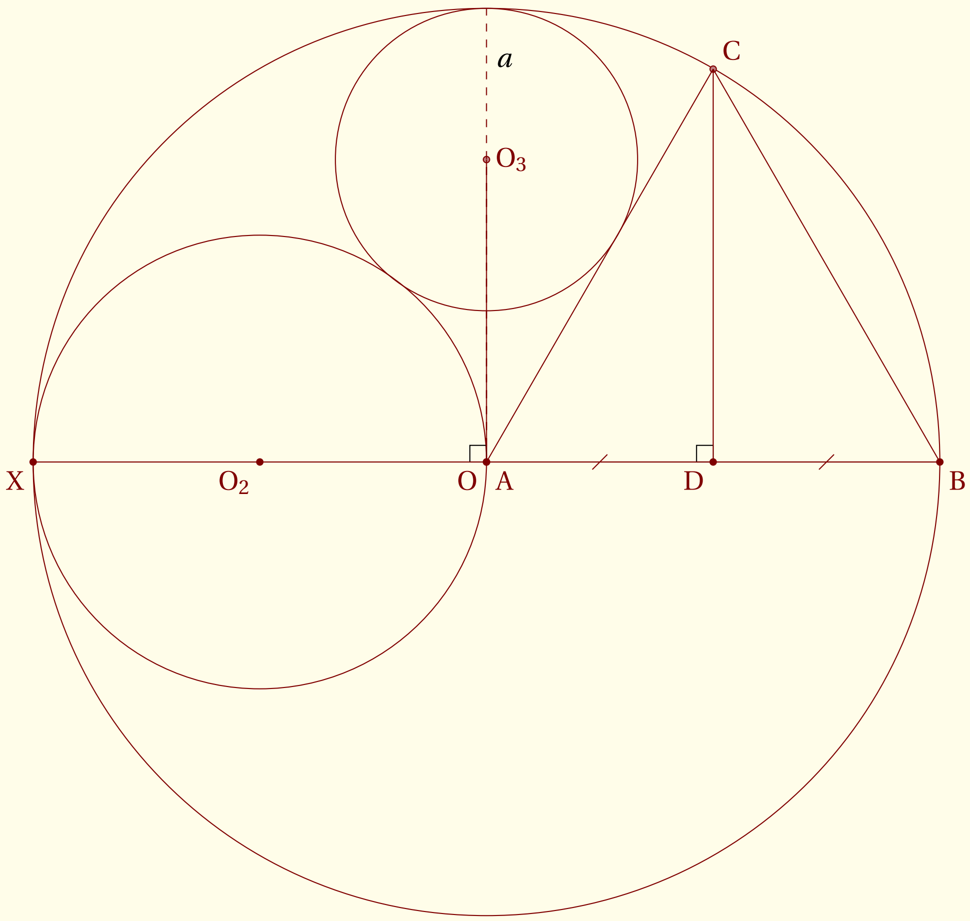
```
\newcommand*{\CircleInscribing}[2]{%
\xdef\ORadius{#1}
\xdef\OORadius{#2}
\pgfmathparse{(2*(\ORadius-\OORadius))/(\ORadius/\OORadius+1)}%
\global\let\OOORadius\pgfmathresult%
\pgfmathparse{\ORadius-\OOORadius}%
\global\let\OOOORadius\pgfmathresult%
\pgfmathparse{2*\OORadius-\ORadius}%
\global\let\XA\pgfmathresult%
\tkzPoint[pos = below left](0,0){O}
\ifdim\XA pt = 0pt\relax%
\tkzPoint[pos = below right](\XA,0){A}
\else
\tkzPoint[pos = below left](\XA,0){A}
\fi
\tkzPoint[pos = below left](\OORadius,0){D}
\tkzPoint[pos = below left](-\ORadius,0){X}
\tkzPoint[pos = below right](\ORadius,0){B}
\tkzPoint[name = $O_2$,pos = below left](\OORadius-\ORadius,0){O2}

\tkzDefLine[mediator](A,B)             \tkzGetPoints{mr}{ml}
\tkzInterLC[R](D,mr)(O,\ORadius cm)      \tkzGetPoints{C}{E}
\tkzDefLine[orthogonal=through A](X,A) \tkzGetPoint{pr}
 \ifdim\XA pt < 0 pt\relax
    \tkzInterLC[R](A,pr)(O,\OOOORadius cm) \tkzGetPoints{O4}{O3}
    \else
\ifdim\XA pt = 0pt\relax
 \tkzInterLC[R](A,pr)(O,\OOOORadius cm) \tkzGetPoints{O4}{O3}
 \else
    \tkzInterLC[R](A,pr)(O,\OOOORadius cm) \tkzGetPoints{O3}{O4}
 \fi   \fi
  \tkzInterLC[R](O,O3)(O,\ORadius cm)  \tkzGetPoints{W}{Z}
 \tkzDrawSegments(D,B D,A)
 \tkzMarkSegments[mark = s|](D,B D,A)
 \tkzDrawCircle[R](O,\ORadius cm)
 \tkzDrawCircle[R](O2,\OORadius cm)
 \tkzMarkRightAngles(X,D,C X,A,O3)
 \tkzDrawCircle[R](O3,\OOORadius cm)
 \tkzDrawPoints(O3,C)
 \tkzLabelPoint[right](O3){$O_3$}
 \tkzLabelPoints[above right](C)
 \tkzDrawSegment[style = dashed](O,O3)
 \tkzDrawSegments(A,O3 C,B C,A X,B C,D)
 \tkzDrawSegment[style = dashed](O,Z)
 \tkzLabelSegment[pos=.85,above right](O,Z){$a$}
}
```



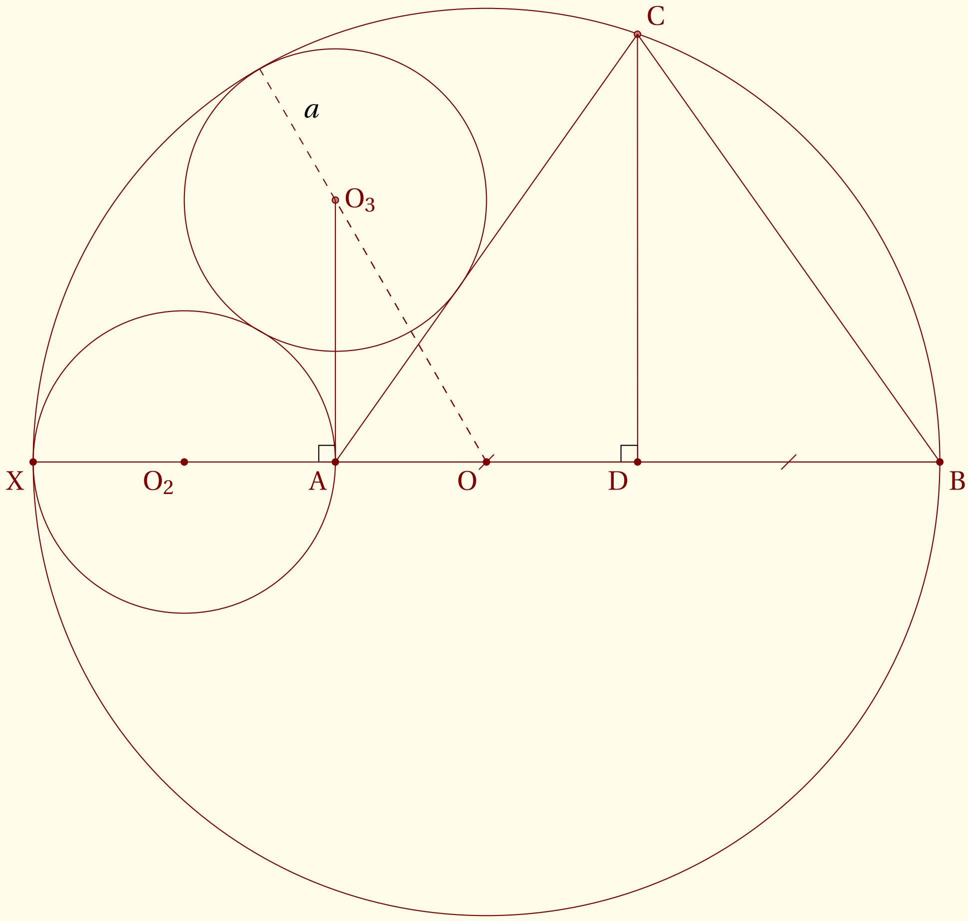

```
\begin{tikzpicture}[scale = 1]
   \tkzInit[xmin = -7,xmax = 7,ymin = -7,ymax = 7]
   \tkzClip
   \CircleInscribing{6}{4}
\end{tikzpicture}
```

Figure 1: Sangaku problem (1803) : R1 = 6 cm et R2 = 4 cm



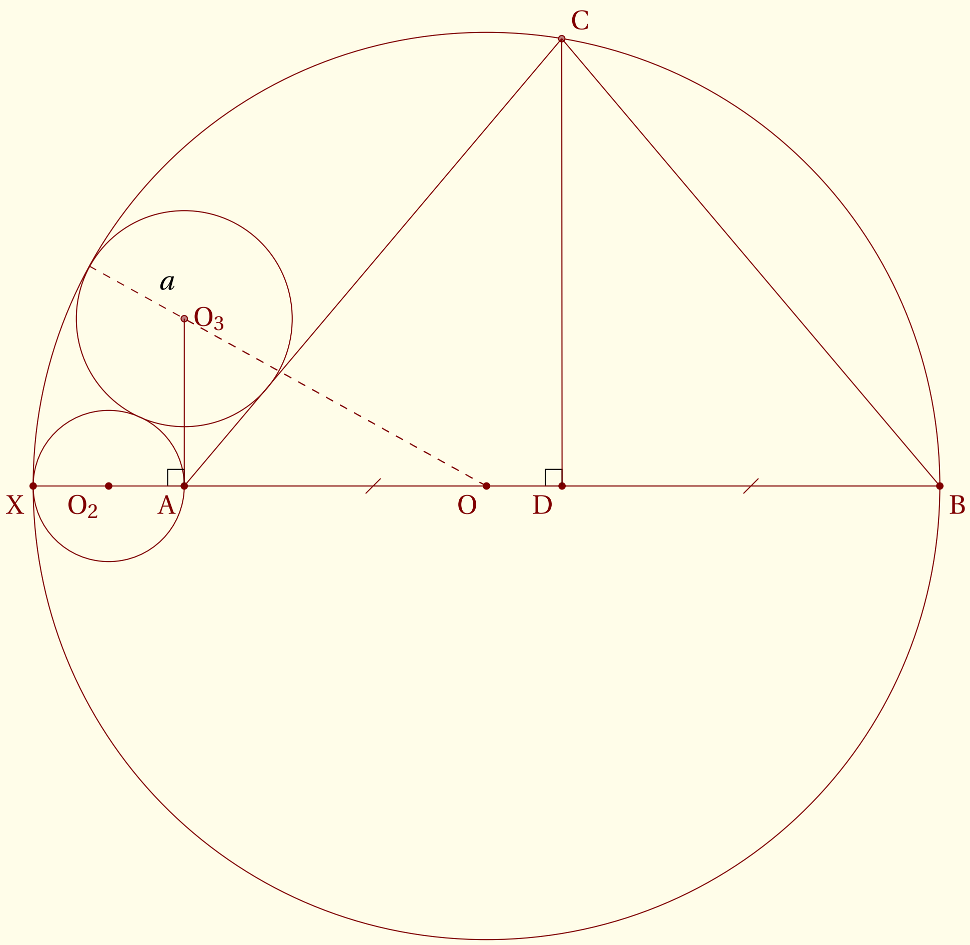

```
\begin{tikzpicture}[scale = 1]
   \tkzInit[xmin = -7,xmax = 7,ymin = -7,ymax = 7]
   \tkzClip
   \CircleInscribing{6}{3}
 \end{tikzpicture}
```

Figure 2: Sangaku problem (1803) : R1 = 6 cm et R2 = 3 cm




```
\begin{tikzpicture}[scale = 1]
   \tkzInit[xmin = -7,xmax = 7,ymin = -7,ymax = 7]
   \tkzClip
   \CircleInscribing{6}{2}
\end{tikzpicture}
```

Figure 3: Sangaku problem (1803) : R1 = 6 cm et R2 = 2 cm




```
\begin{tikzpicture}[scale = 1]
   \tkzInit[xmin = -7,xmax = 7,ymin = -7,ymax = 7]
   \tkzClip
   \CircleInscribing{6}{1}
\end{tikzpicture}
```

Figure 4: Sangaku problem (1803) : R1 = 6 cm et R2 = 1 cm

SECTION 7

Sangaku - Harmonic mean

7.1 Harmonic mean of two numbers

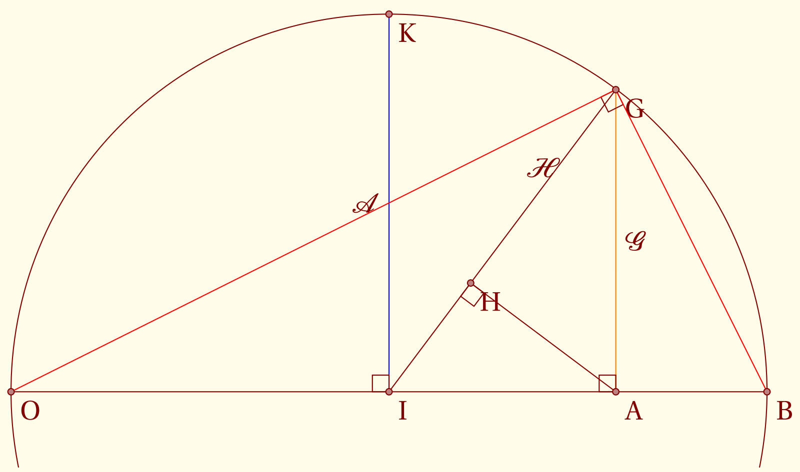
$$\frac{2}{c} = \frac{1}{a} + \frac{1}{b}$$

which says that c is the harmonic mean of a and b.
For two numbers a and b, $\mathscr{A} = \frac{a+b}{2}$ and $\mathscr{G} = \sqrt{ab}$, and $\mathscr{H}$ such as $\frac{2}{\mathscr{H}} = \frac{1}{a} + \frac{1}{b}$

- a and b two numbers such as OA $= a$ and AB $= b$. I is the center of the circle $\mathscr{C}$ with diameter [OB]. [IK] is a radius perpendicular to (OB). It's easy to prove IK $= \mathscr{A}$.
- (AG) is a line perpendicular to (OB) in A and G is point of $\mathscr{C}$. OGB is a right triangle, so

$$\mathrm{AG}^2 = \mathrm{OA} \times \mathrm{OB}$$

Finally we get AG $= \mathscr{G} = \sqrt{ab}$



GAH and IAG are right rectangles and these rectangles are similars so :

$$\frac{\mathrm{GH}}{\mathrm{AG}} = \frac{\mathrm{AG}}{\mathrm{IH}} \text{ and } \frac{\mathrm{AG}}{\mathscr{G}} = \frac{\mathscr{G}}{\mathscr{A}}$$

In this case, we have

$$\mathscr{G}^2 = \mathscr{A} \times \mathrm{AG} \text{ and } \mathrm{AG} = \frac{\mathscr{G}^2}{\mathscr{A}}$$

Now we can prove AG $= \mathscr{H}$

$$\frac{1}{\mathscr{H}} = \frac{\mathrm{A}}{\mathscr{G}^2} = \frac{a+b}{2ab}$$

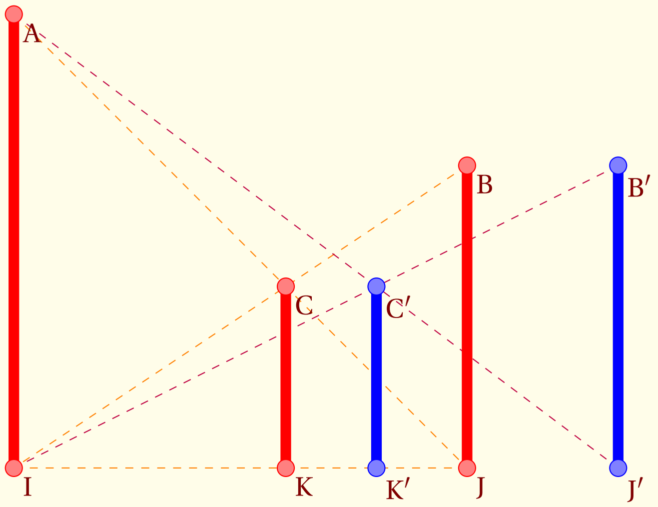
Finally

$$\frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) = \frac{1}{\mathrm{H}}$$

7.2 Sangaku - Harmonic mean

Two vertical segments AI, BJ *(*AI = a *and* BJ = b*), the intersection* C *of the diagonals is at the height that depends solely on* AI *and* BJ. *In fact*

$$\frac{1}{\mathrm{CK}} = \frac{1}{\mathrm{AI}} + \frac{1}{\mathrm{BJ}} = \frac{1}{a} + \frac{1}{b}$$




```
\begin{tikzpicture}
  \tkzInit[xmin = -1,xmax = 9,ymax = 7]\tkzClip[space=.5]
  \tkzDefPoint(0,6){A}\tkzDefPoint(0,0){I}
  \tkzDefPoint(6,4){B}\tkzDefPoint(8,4){B'}
  \tkzDefPoint(6,0){J}\tkzDefPoint(8,0){J'}
  \tkzInterLL(A,J)(B,I)   \tkzGetPoint{C}
  \tkzDefPointBy[projection=onto I--J](C) \tkzGetPoint{K}
  \tkzInterLL(A,J')(B',I) \tkzGetPoint{C'}
  \tkzDefPointBy[projection=onto I--J](C')  \tkzGetPoint{K'}
  \tkzDrawSegments[color = orange,style=dashed](A,J B,I I,J)
  \tkzDrawSegments[color = purple,style=dashed](A,J' B',I)
  \tkzDrawSegments[line width = 4pt,color = red](C,K A,I B,J)
  \tkzDrawSegments[line width = 4pt,color = blue](C',K' B',J')
  \tkzDrawPoints[color = red,size = 16](A,I,C,K,B,J)
  \tkzDrawPoints[color = blue,size = 16](C',K',B',J')
  \tkzLabelPoints(A,I,C,K,B,J,C',K',B',J')
\end{tikzpicture}
```

Let's denote AI = a, BJ = b, CK = c, IK = α and KJ = β.

The triangles AIJ and CKJ are similar as are triangles BJI and CKI. We have the proportion

$$\frac{c}{a} = \frac{\beta}{\alpha+\beta} \text{ and } \frac{c}{b} = \frac{\alpha}{\alpha+\beta}$$

We can add the two equalities

$$\frac{c}{a} + \frac{c}{b} = \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta} = 1$$

A division by c gives the desired result:

$$\frac{1}{c} = \frac{1}{a} + \frac{1}{b}$$

which says that c is the half of the harmonic mean of a and b.

SECTION 8

Two Unrelated Circles

Some references

- a Scientific American article by Tony Rothman written in co-operation with Hidetoshi Fukagawa;
- the book by H. Fukagawa and D. Pedoe.
- http://www.sangaku.info/
- http://mathworld.wolfram.com
- http://www.wasan.jp/english/
- http://www.cut-the-knot.org/pythagoras/Sangaku.shtml

8.1 Sangaku Two Unrelated Circles

Chord [ST] *is perpendicular to diameter* [CP] *of a circle with center* O *at point* R. Q *is point of* [CP] *between* P *and* R . [SQ] *intersects the circle in* V.
Chords [KN] *and* [ST] *are perpendicular to diameter* [CP] *of a circle with center* O *at points* Q *and* R. [SQ] *intersects the circle in* V. *(*K, S *are on one side of* [CP], N *and* T *on the other.* Q *is between* P *and* R.*) Let* r *be the radius of the circle inscribed into the curvilinear triangle* TQV. *Prove that*

$$\frac{1}{r} = \frac{1}{PQ} + \frac{1}{QR}$$

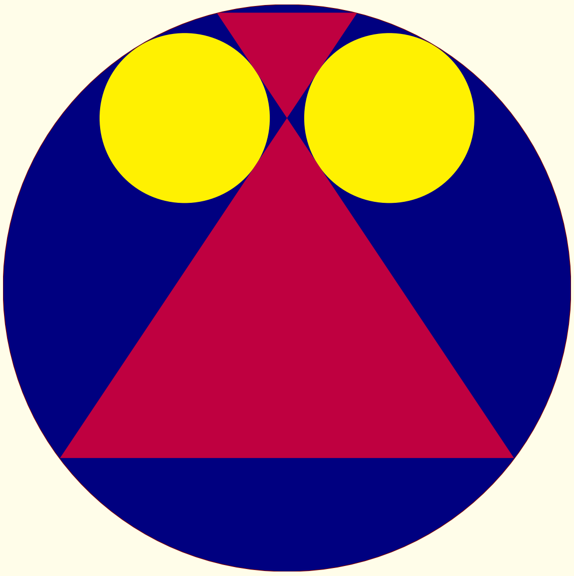


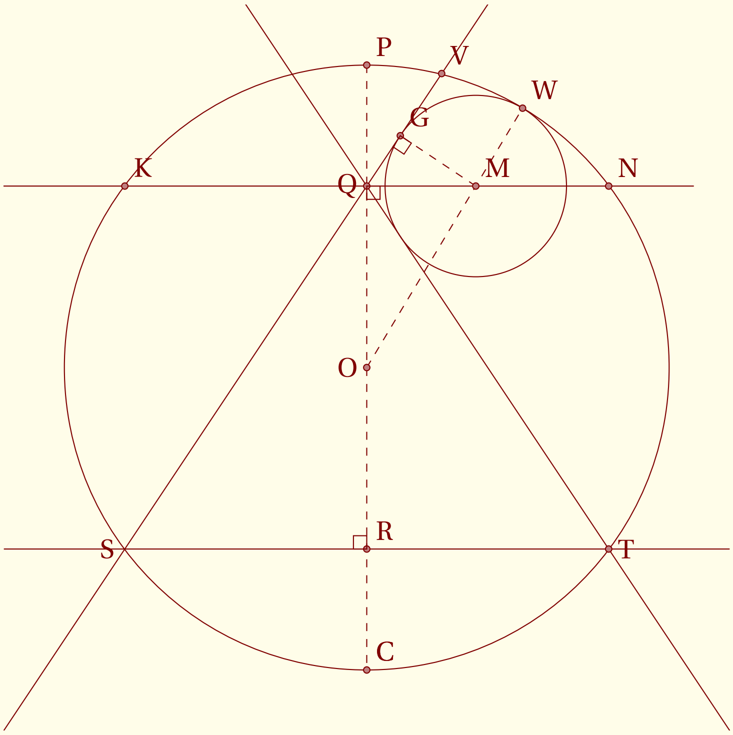
Figure 5: Sangaku Two Unrelated Circles

Now you can read the code to get the last picture.

```
\begin{tikzpicture}[scale=.75]
    \tkzInit[xmin = -5,ymin = -5,xmax = 5,ymax = 5] \tkzClip
    \tkzDefPoint(0,0){O}  \tkzDefPoint(-2,-3){A}
    \tkzDefPoint(2,-3){B} \tkzDefPoint(0,3){Q}
    \tkzDrawCircle[R](O,5 cm)
    \tkzInterLC[R](A,B)(O,5 cm)
        \tkzGetPoints{S}{T}
    \tkzInterLC[R](S,Q)(O,5 cm)
            \tkzGetSecondPoint{X}
    \tkzInterLC[R](T,Q)(O,5 cm)
            \tkzGetFirstPoint{Z}
    \tkzDefMidPoint(S,T) \tkzGetPoint{R}
    \tkzDefLine[orthogonal=through Q](O,Q)
    \tkzCalcLength(R,Q)\tkzGetLength{dRQ}
    \tkzCalcLength(S,Q)\tkzGetLength{dSQ}
    \tkzDefPoint(\dSQ/\dRQ*1.5,3){M}
    \tkzDefPoint(-\dSQ/\dRQ*1.5,3){N}
    \tkzDefPointBy[projection=onto S--Q](M)
       \tkzGetPoint{G}
    \tkzInterLC[R](O,M)(O,5 cm)
        \tkzGetPoints{W}{Y}
    \tkzClipCircle[R](O,5 cm)
    \tkzDrawLines[add= 1 and 1](A,B S,Q T,Q)
    \tkzDrawCircle[R](M,1.5 cm)
    \tkzFillCircle[R,color = blue!50!Black](O,5cm)
    \tkzFillCircle[R,color = yellow](M,1.5cm)
    \tkzFillCircle[R,color = yellow](N,1.5cm)
    \tkzFillPolygon[color = purple](S,T,Q)
    \tkzFillPolygon[color = purple](Z,X,Q)
\end{tikzpicture}
```

8.2 Explanation : Idea of Nathan Bowler

Take coordinates such that it is the unit circle ($r = 1$) with origin M and with Q on the x axis (Q is between O and P). Let G = $(a\,;\,b)$, W = $(u\,;\,v)$, M = (0 ; 0) with $a < 0$ and $u > 0$
Then:



- The line (QS) has equation $ax+by=1$, (MG = 1 and (MG) perpendicular to (QS))
- The line (MW) has equation $xv-yu=0$,
- Q is at $\left(\frac{1}{a}\;;\;0\right)$,
- O is at $\left(\frac{1}{a}\;;\;\frac{v}{au}\right)$,
- $\text{OQ}=-\frac{v}{au}$,
- $\text{OW}^2=\left(1-\frac{1}{au}\right)^2$

$$\begin{aligned}\text{OW}^2 &= \left(u-\frac{1}{a}\right)^2+\left(v-\frac{v}{au}\right)^2\\ &= u^2-\frac{2u}{a}+\frac{1}{a^2}+v^2-\frac{v^2}{u^2a^2}-\frac{2v^2}{ua}\\ &= 1-\frac{2u}{a}+\frac{1}{a^2}-\frac{1-u^2}{u^2a^2}-\frac{2-2u^2}{ua}\\ &= \left(1-\frac{1}{ua}\right)^2.\end{aligned}$$

- S and V are points of circles with center O and M, so if $(x\,;\,y)$ are coordinates of these points $u^2+v^2=1$ and $\left(u-\frac{1}{a}\right)^2+\left(v-\frac{v}{au}\right)^2=\left(1-\frac{1}{ua}\right)^2$
  Further, we obtain
  $$y^2-\frac{av}{u}2y+2\frac{a}{u}-a^2-1=0$$
  The discriminant is $\left(\frac{a^2}{u^2}-2\frac{a}{u}+1\right)^2=\left(\frac{a}{u}-1\right)^2=\left(1-\frac{a}{u}\right)^2$
  This equation has two solutions :
  $y_1=\frac{av}{u}-\left(1-\frac{a}{u}\right)$ negative and $y_2=\frac{av}{u}+\left(1-\frac{a}{u}\right)$ positive.

Finally

$$\begin{aligned}\text{QR}&=-y_1\\&=1-\frac{a}{u}-\frac{av}{u}.\end{aligned}$$

and

$$\begin{aligned}\text{PQ}&=\text{OP}-\text{OQ}=\text{OW}-\text{OQ}=1-\frac{1}{ua}+\frac{v}{au}\\\text{PQ}&=1-\frac{1}{au}+\frac{v}{au}=1-\frac{1-v}{au}.\end{aligned}$$

So

$$(\text{PQ}-1)(\text{QR}-1)=\left(-\frac{a}{u}-\frac{av}{u}\right)\left(-\frac{1-v}{au}\right)=\frac{(1-v)(1+v)}{u^2}=1.$$

Equivalently,

$$\frac{1}{\text{PQ}}+\frac{1}{\text{QR}}=1=\frac{1}{r}.$$

Figure 6: Sangaku-Two Unrelated Circles : R = 6 cm, OR = −2 cm et RQ = 4 cm

```
\begin{tikzpicture}[scale = .5]
   \TwoUnrelatedCircles{6}{-2}{4}
\end{tikzpicture}
```

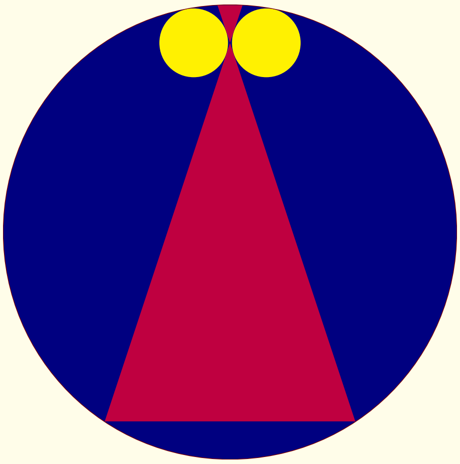
In the general case, some information will be needed. For instance, it is necessary to give the radius of the big circle, the position of points R and Q. I have decided to give R$(0, r)$ relatively to O and Q relatively to P with the value of PQ.

8.3 A macro \TwoUnrelatedCircles

The macro, below, is used to obtain some examples.

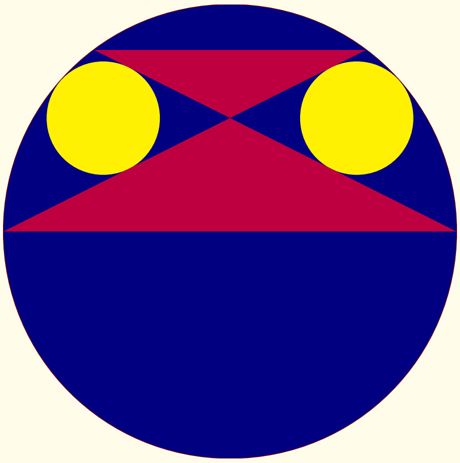
```
\newcommand*{\TwoUnrelatedCircles}[3]{
 % #1 --> R ; #2 --> Y_P <0 or >0 ; #3 --> PQ > =  0
 \xdef\ORadius{#1} \xdef\tucR{#2} \xdef\tucQR{#3}
 \pgfmathparse{\tucQR+\tucR} \xdef\tucQ{\pgfmathresult}
 \pgfmathparse{\ORadius-(\tucQR+\tucR)} \xdef\tucPQ{\pgfmathresult}
 \pgfmathparse{(\tucPQ*\tucQR)/(\tucPQ+\tucQR)} \let\tucr\pgfmathresult
  \pgfmathparse{\ORadius-\tucr} \let\OORadius\pgfmathresult%
 \tkzInit[xmin = -\ORadius,ymin = -\ORadius,xmax = \ORadius, ymax = \ORadius]
 \tkzClip
 \tkzDefPoint(0,0){O}            \tkzDefPoint(-\ORadius,\tucR){A}
 \tkzDefPoint(\ORadius,\tucR){B} \tkzDefPoint(0,\tucQ){Q}
 \tkzInterLC[R](A,B)(O,\ORadius cm)  \tkzGetPoints{S}{T}
 \tkzInterLC[R](S,Q)(O,\ORadius cm)  \tkzGetPoints{S}{V}
 \tkzInterLC[R](Q,T)(O,\ORadius cm)  \tkzGetPoints{Y}{T}
 \tkzDefPoint(0,\tucR){R}
 \tkzDefPoint(-\ORadius,\tucQ){U}
 \tkzInterLC[R](U,Q)(O,\OORadius cm) \tkzGetPoints{M}{N}
 \tkzDrawCircle[R](O,\ORadius cm)
 \tkzClipCircle[R](O,\ORadius cm)
 \tkzDrawLines[add= 1 and 1](S,Q T,Q)
 \tkzDrawCircle[R](M,\tucr cm)
 \tkzFillCircle[R,color = blue!50!Black](O,\ORadius cm)
 \tkzFillCircle[R,color = yellow](M,\tucr cm)
 \tkzFillCircle[R,color = yellow](N,\tucr cm)
 \tkzFillPolygon[color = purple](S,T,Q)
 \tkzFillPolygon[color = purple](V,Y,Q)
 }
```

New examples



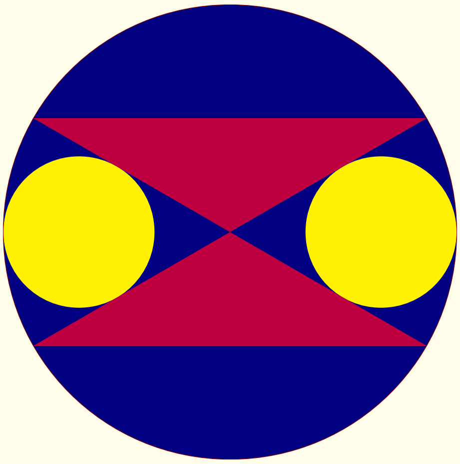
```
\begin{tikzpicture}[scale = .5]
   \TwoUnrelatedCircles{6}{-5}{10}
 \end{tikzpicture}
```

Figure 7: Sangaku-Two Unrelated Circles : R = 6 cm, OR = −5 cm et RQ = 10 cm



```
\begin{tikzpicture}[scale = .5]
   \TwoUnrelatedCircles{6}{0}{3}
 \end{tikzpicture}
```

Figure 8: Sangaku-Two Unrelated Circles : R = 6 cm, OR = 0 cm et RQ = 3 cm



```
\begin{tikzpicture}[scale = .5]
   \TwoUnrelatedCircles{6}{-3}{3}
 \end{tikzpicture}
```

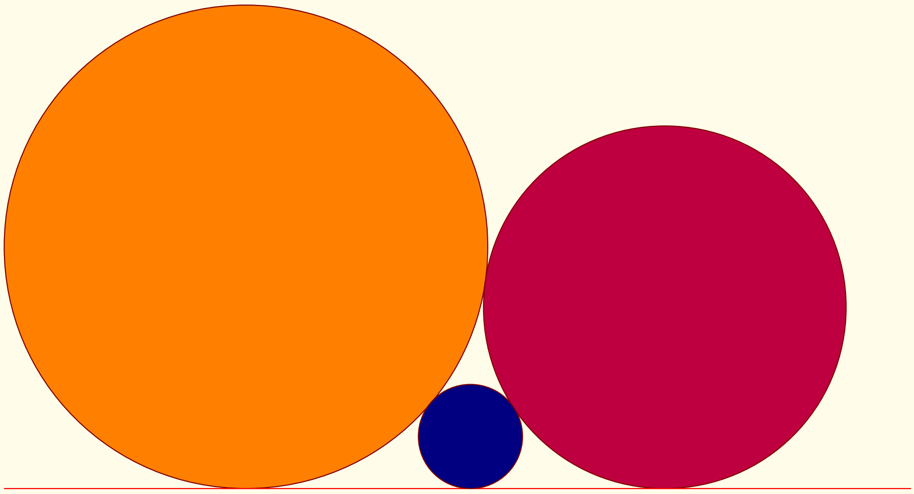
Figure 9: Sangaku-Two Unrelated Circles : R = 6 cm, OR = −3 cm et RQ = 3 cm

SECTION 9

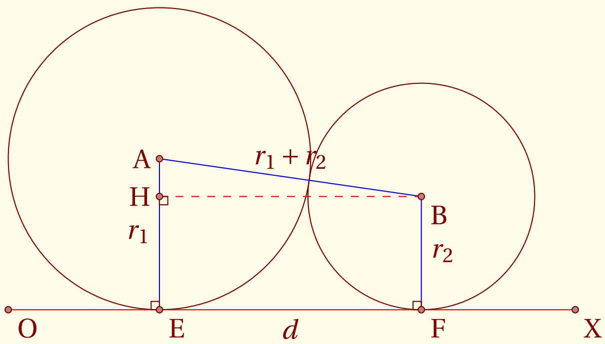
Sangaku - Three Tangent Circles

Given three circles tangent to each other and to a straight line, express the radius of the middle circle via the radii of the other two. This problem was given as a Japanese temple problem on a tablet from 1824 in the Gumma Prefecture (MathWorld)

9.1 Example Three Tangent Circles



```
\begin{tikzpicture}[scale=.8]
  \tkzInit[xmin = -1,ymin = -1,xmax = 16]\tkzClip
  \tkzDefPoint(4,4){A}\tkzDefPoint(10.928,3){B}
  \tkzDefPoint(0,0){O}\tkzDefPoint(15,0){X}
  \tkzDrawSegment[color = red](O,X)
  \tkzDrawCircle[R,fill = orange](A,4 cm)
  \tkzDrawCircle[R,fill = purple](B,3 cm)
  \pgfmathparse{4+8*sqrt(3)/(2+sqrt(3))}
  \edef\cx{\pgfmathresult}
  \pgfmathparse{12/((2+sqrt(3))*(2+sqrt(3)))}
  \edef\cy{\pgfmathresult}
  \tkzDefPoint(\cx,\cy){C}
  \tkzDrawSegment[color = red](O,X)
  \tkzDrawCircle[R,fill = blue!50!black](C,\cy cm)
\end{tikzpicture}
```



9.2 Explanation

Lemma : Given two circles tangent to each other and to a straight line at points E and F, we can express EF with the radii (r_1 , r_2) of the two circles :

$$\mathrm{EF} = 2\sqrt{r_1 r_2}$$

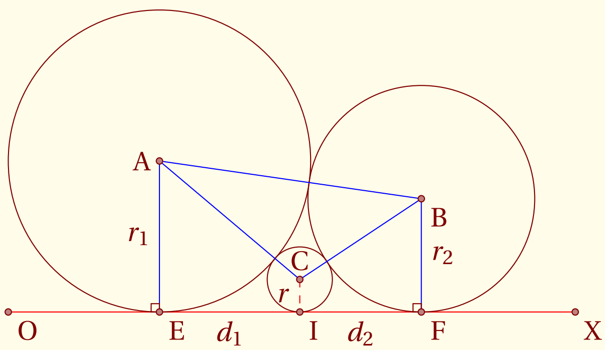
Proof : ABH is a right triangle with hypotenuse AB = $r_1 + r_2$. We have by the Pythagorean theorem

$$\begin{aligned}\mathrm{AB}^2 &= \mathrm{AH}^2 + \mathrm{HB}^2\\ &= \mathrm{AH}^2 + d^2\end{aligned}$$

If $r_1 > r_2$

$$\begin{aligned}\mathrm{AB}^2 &= \mathrm{AH}^2 + \mathrm{HB}^2\\ (r_1 + r_2)^2 &= (r_1 - r_2)^2 + d^2\end{aligned}$$

Finally $d^2 = 4r_1 r_2$ and $d = 2\sqrt{r_1 r_2}$.
Now we have two circles ($r_1 > r_2$) tangent to each other and to a straight line, and we want to draw another circle tangent to each others and to the same straight line :



We can apply the lemma to the circles (A, r_1) and (B, r_2), then to the circles (A, r_1) and (C, r) and finally to the circles (B, r_2) and (C, r).
We get four equations :

$$
\begin{aligned}
d &= d_1 + d_2 \\
d &= 2\sqrt{r_1 r_2} \\
d_1 &= 2\sqrt{r_1 r} \\
d_2 &= 2\sqrt{r r_2}
\end{aligned}
$$

After simplification, the equations become

$$2\sqrt{r_1 r_2} = 2\sqrt{r_1 r} + 2\sqrt{r r_2}$$

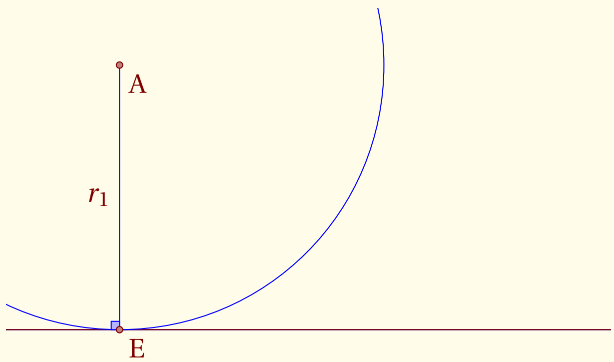
Divide now by $\sqrt{r}\sqrt{r_1}\sqrt{r_2}$ to obtain

$$\frac{1}{\sqrt{r}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$$

This is the condition between the radii.

9.3 How to make this construction with a ruler and a compass

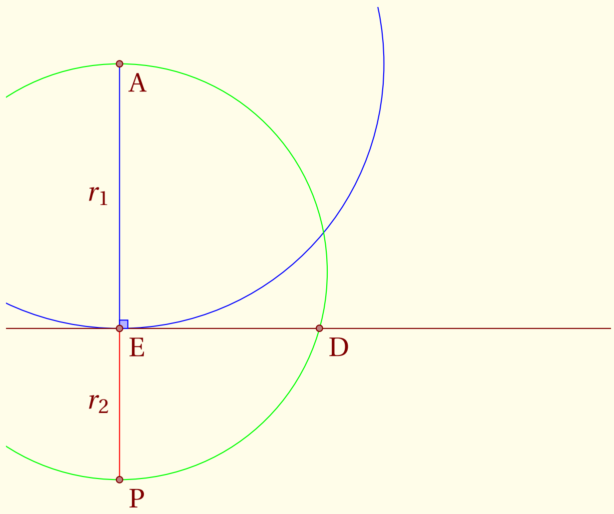
step 1. We draw a line (OX) and a circle with center A tangent to the line in E. The radius of this circle is EA = r_1 = 7 cm.



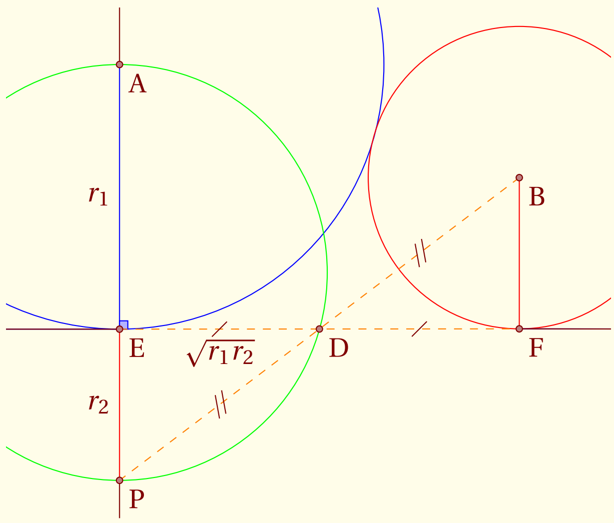
step 2. Now we want to draw a second circle with center B of radius r_2 = FB = 4 cm mutually tangent to the first circle and to the line (OX) in F.

We place a point P on the line (EA) such as AP = r_2 = 4 cm

The circle with diameter AP intercepts the (OX) axis in a point D such as the length OD = $\sqrt{r_1 r_2}$.



AED and DEP are right similar triangles and we can write $\frac{ED}{EA} = \frac{EP}{ED}$. It is now easy to obtain F such EF = 2ED = $2\sqrt{r_1 r_2}$. B is a point such as B and P are symmetric with D point of symmetry. We can draw the circle with center B and radius BF.



step 3. Now we can use the sangaku about harmonic mean. The line EB intercepts the line AF in a point K such as

$$\frac{1}{KJ} = \frac{1}{EA} + \frac{1}{FB} = \frac{1}{r_1} + \frac{1}{r_2}$$

$$\frac{1}{\sqrt{r}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$$

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{2}{\sqrt{r_1 r_2}}$$

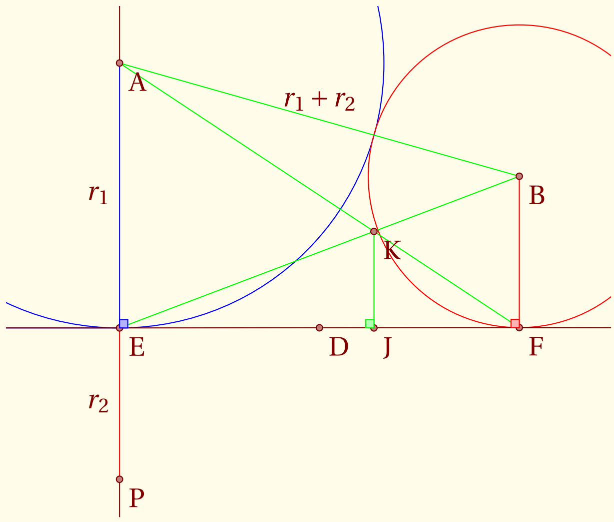
$$\frac{1}{r} = \frac{1}{KJ} + \frac{1}{\frac{\sqrt{r_1 r_2}}{2}}$$

To find r , we need only to represent $\frac{\sqrt{r_1 r_2}}{2}$.

OD $= \sqrt{r_1 r_2}$, OM $= \frac{\sqrt{r_1 r_2}}{2}$ and ON = OM

step 4. if I is the projection of R on (EF) axis, we have IR $= r$. Let S the projection point of R on (EA) axis.

AB $= r_1 + r_2$ et les cercles (A, r_1) et (B, r_2) sont tangents. Les droites (A, F) et (E, B) se coupent en un point K. Soit J la projection orthogonale de K sur (E, F). Le sangaku précédent en relation avec la moyenne harmonique permet d'écrire que



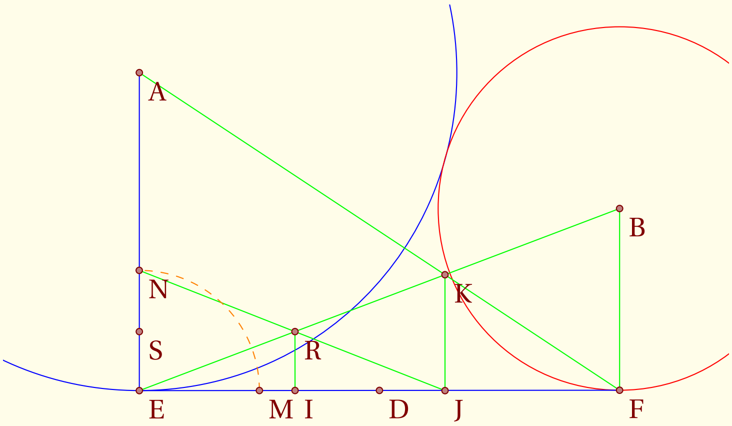
$$\frac{1}{KJ} = \frac{1}{EA} + \frac{1}{FB} = \frac{1}{r_1} + \frac{1}{r_2}$$

From $\frac{1}{\sqrt{r}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$ we get :

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{2}{\sqrt{r_1 r_2}} = \frac{1}{KJ} + \frac{1}{\frac{\sqrt{r_1 r_2}}{2}}$$

On the next picture, $\mathrm{EM} = \mathrm{EN} = \dfrac{\mathrm{ED}}{2} = \dfrac{\sqrt{r_1 r_2}}{2}$. The line (EK) intercepts the line (JN) at the point R and $\mathrm{IR} = r$. Again, we can use the sangaku about harmonic mean.

$$\frac{1}{\mathrm{IR}} = \frac{1}{\mathrm{KJ}} + \frac{1}{\mathrm{EN}}$$



step 5. The point C is the intersection of the circle with center A and radius $a + r$ and the line (SR). In the next picture, $\mathrm{ES} = \mathrm{ET}$. C is the intersection of the circle $(\mathrm{A}, \mathrm{AT})$ and the line (SR).