

Alain Matthes

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<http://altermundus.fr>

Sangakus (rule and compass)

Alain Matthes

This document is a little gallery of some simple sangakus. I made all these pictures with my package `tkz-euclide.sty v2.55c`. Sangaku or San Gaku are colorful wooden tablets which were hung often in shinto shrines and sometimes in buddhist temples in Japan and posing typical and elegant mathematical problems. The problems featured on the sangaku are problems of japanese mathematics (wasan). The earliest sangaku found date back to the beginning of the 17th century.

☞ I would like to thank **Till Tantau** for the beautiful LATEX package, namely TikZ.

☞ Vous trouverez la plupart des exemples de ce document et de nombreux autres exemples sur mon site : altermundus.fr

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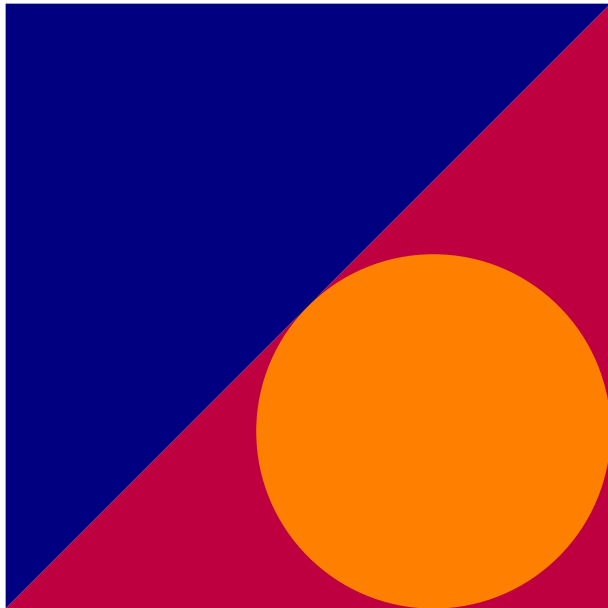
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1 Sangaku in a square - A very simple sangaku.

1.1 The picture

Find a relationship between the radius of the yellow circle and the side of the square.



1.2 Explanation

Firstly, we can find the relationship between the inradius and the sides of a right triangle.

If r is the inradius of a circle inscribed in a right triangle with sides a and b and hypotenuse c , then

$$r = \frac{1}{2}(a + b - c).$$

Let ABC represents a right triangle, with the right angle located at C , as shown on the figure. Let a , b and c the lengths of the three sides; c is the

length of the hypotenuse.

Let r and p be the radius of the incircle and the semiperimeter of the triangle.

a , b and c can be regarded in relation to r and they may be expressed with r : $a = r + (a - r)$, $b = r + (b - r)$ and $c = (a - r) + (b - r)$.

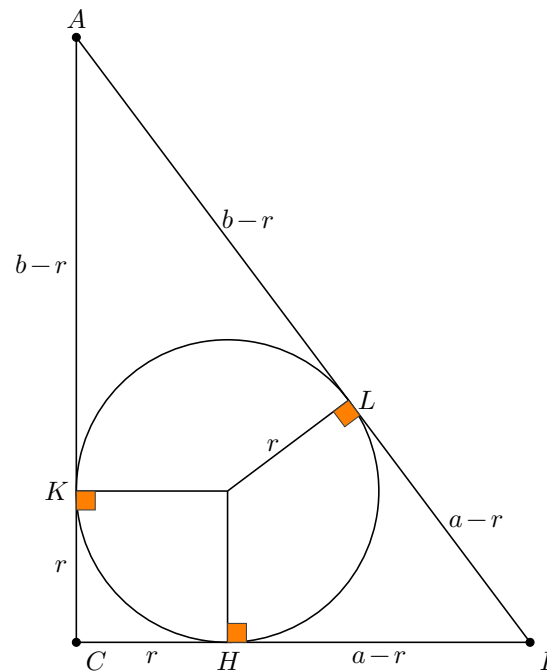
In a right triangle, we have the relation $r = p/2 - c$. From the diagram, the hypotenuse AB is split in two pieces: $(a - r)$ and $(b - r)$, the length of the hypotenuse is $c = (a - r) + (b - r)$.

The perimeter is a function of r

$$p = a + b + c = r + (a - r) + r + (b - r) + (a - r) + (b - r) = 2a + 2b - 2r$$

so we can express r with s and c

$$2r = a + b - c = p - 2c \text{ and } r = \frac{p}{2} - c = \frac{a + b - c}{2}$$



```

\begin{tikzpicture}
\tkzDefPoint(0,8){A} \tkzDefPoint(6,0){B}
\tkzDefPoint(0,0){C} \tkzLabelPoints(B,C)
\tkzDrawPolygon(A,B,C)
\tkzInCenter(A,B,C) \tkzGetPoint{I}
\tkzDefPointBy[projection=onto B--C](I)
\tkzGetPoint{H}
\tkzDefPointBy[projection=onto A--C](I)
\tkzGetPoint{K}
\tkzDefPointBy[projection=onto A--B](I)
\tkzGetPoint{L}
\tkzDrawSegments(I,L I,H I,K)
\tkzDrawCircle(I,H)
\tkzMarkRightAngles%
[fill=orange](I,L,B B,H,I C,K,I)
\tkzLabelSegment[below](C,H){$r$}
\tkzLabelSegment[below](H,B){$a-r$}
\tkzLabelSegment[left](C,K){$r$}
\tkzLabelSegment[left](K,A){$b-r$}
\tkzLabelSegment[right](L,B){$a-r$}
\tkzLabelSegment[right](A,L){$b-r$}
\tkzLabelSegment[left](I,L){$r$}
\tkzLabelPoint[below](H){$H$}
\tkzLabelPoint[left](K){$K$}
\tkzLabelPoint[right](L){$L$}
\tkzDrawPoints(A,B,C)
\tkzLabelPoint[above](A){$A$}
\end{tikzpicture}

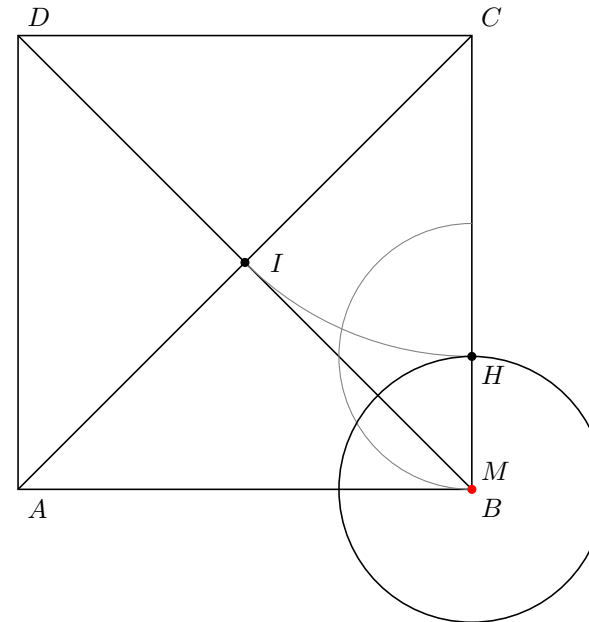
```

Now, let ABC represents a isosceles right triangle with $AB = AC = a$, then $BC = \sqrt{2}a$ and $a + b - c = 2a - \sqrt{2}a$

So the inradius in this case is

$$r = \frac{2 - \sqrt{2}}{2} a$$

Now we can obtain the incenter without the bisectors



```

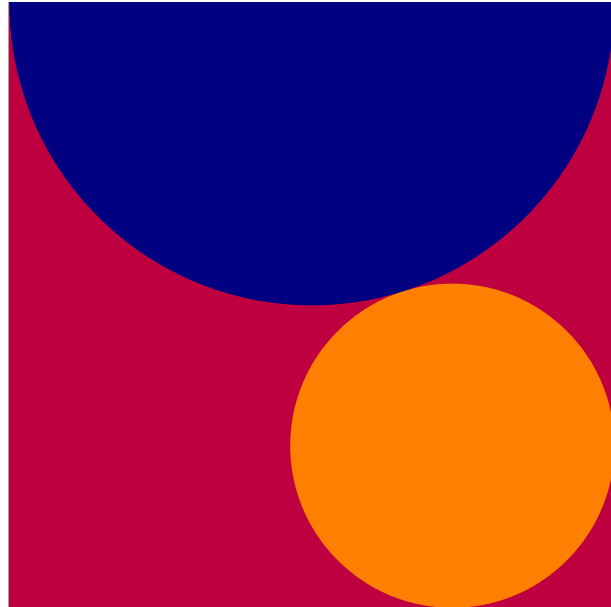
\begin{tikzpicture}[scale=.75]
\tkzDefPoint(0,0){A} \tkzDefPoint(8,0){B}
\tkzDefSquare(A,B)\tkzGetPoints{C}{D}
\tkzDefMidPoint(A,C) \tkzGetPoint{I}
\tkzDuplicateSegment(C,I)(C,B)\tkzGetPoint{H}
\tkzInterLC(B,D)(H,B) \tkzGetPoints{M}{N}
\tkzDrawPolygon(A,B,C,D)
\tkzDrawSegments(A,C B,D)
\tkzDrawArc(C,I)(H)
\tkzDrawArc[rotate](H,B)(-180)
\tkzDrawCircle(M,H)
\tkzDrawPoints(I,H)
\tkzDrawPoint[color = red](M)
\tkzLabelPoints(A,B,H)
\tkzLabelPoints[right=6pt](I)
\tkzLabelPoints[above right](C,D,M)
\end{tikzpicture}

```

2 Sangaku in a square - Circle and semicircle

Find a relationship between the radius of the yellow circle and the side of the square.

2.1 The picture



```
\begin{tikzpicture}
\tkzDefPoint(0,0){A} \tkzDefPoint(8,0){B}
\tkzDefSquare(A,B) \tkzGetPoints{C}{D}
\tkzDrawPolygon(B,C,D,A) \tkzClipPolygon(A,B,C,D)
\tkzDefPoint(4,8){F}
\tkzDefTriangle[equilateral](C,D) \tkzGetPoint{I}
\tkzDefPointBy[projection=onto C--B](I) \tkzGetPoint{J}
\tkzInterLL(D,B)(I,J) \tkzGetPoint{K}
\tkzDefPointBy[symmetry=center K](B)
\tkzGetPoint{M}
\tkzCalcLength(M,I)\tkzGetLength{dMI} \tkzDrawPoint(I)
\tkzDrawCircle(M,I)
\tkzFillPolygon[purple](A,B,C,D)
\tkzFillCircle[R,orange](M,\dMI pt)
\tkzFillCircle[R,blue!50!black](F,4 cm)%
\end{tikzpicture}
```

2.2 Explanation

FNE is a right triangle with hypotenuse [EF]. We have, $EN^2 + NF^2 = EF^2$ by the Pythagorean theorem. In terms of a and r , the theorem appears as

$$\left(\frac{a}{2} - r\right)^2 + (a - r)^2 = \left(\frac{a}{2} + r\right)^2$$

which is equivalent to

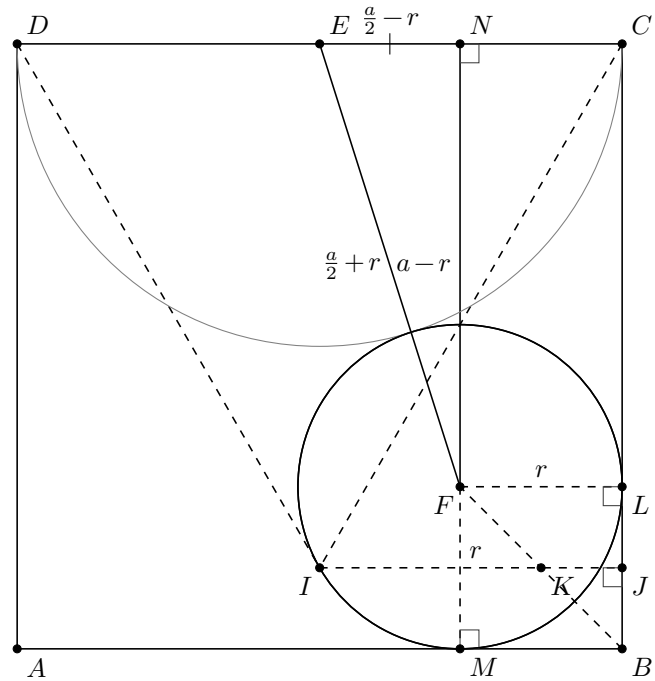
$$r^2 - 4ar + a^2 = r^2 - 4ar + 4a^2 - 3a^2 = (r - 2a - \sqrt{3}a)(r - 2a + \sqrt{3}a) = 0$$

And finally

$$r = a(2 - \sqrt{3}) < a$$

2.3 Construction

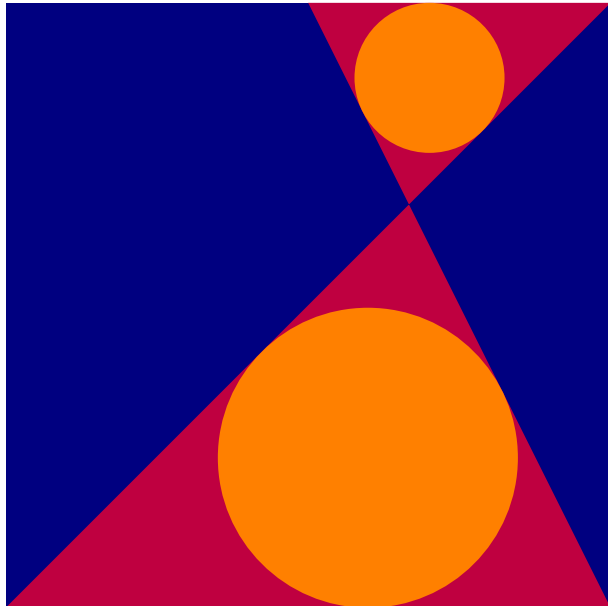
It's easy to prove : $BJ = a(1 - \frac{\sqrt{3}}{2})$ and $BL = a(2 - \sqrt{3})$. First we need to find the point I , then J . K is the intersection of (IJ) and the bisector line of the angle \widehat{ABC} . Finally, F is the symmetric point of B around K .



3 Sangaku in a square - two inscribed circles

In the following diagram, a triangle is formed by a line that joins the base of a square with the midpoint of the opposite side and a diagonal. Find the radius of the two inscribed circles.

3.1 The picture



```

\begin{tikzpicture}
  \tkzDefPoint(0,0){A}
  \tkzDefPoint(8,0){B}
  \tkzDefPoint(4,8){E}
  \tkzDefSquare(A,B) \tkzGetPoints{C}{D}
  \tkzDefLine[bisector](B,A,C) \tkzGetPoint{a}
  \tkzDefLine[bisector](E,B,A) \tkzGetPoint{b}
  \tkzInterLL(A,a)(B,b) \tkzGetPoint{K}
  \tkzDefPointBy[projection=onto A--B](K) \tkzGetPoint{H}
  \tkzDefLine[bisector](D,C,A) \tkzGetPoint{c}
  \tkzDefLine[bisector](B,E,C) \tkzGetPoint{e}
  \tkzInterLL(C,c)(E,e) \tkzGetPoint{K1}
  \tkzDefPointBy[projection=onto C--D](K1) \tkzGetPoint{H1}
  \tkzFillPolygon[color = blue!50!black](A,B,C,D)
  \tkzFillPolygon[color = purple](A,C,E,B)
  \tkzFillCircle[color = orange](K,H)
  \tkzFillCircle[color = orange](K1,H1)
\end{tikzpicture}

```

3.2 Explanation and construction

QC is parallel to the base AB and is half as long which implies that the two triangles QIC and QAB are similar. I divides the segments QP and AD in ratio 2:1 so that

$$IP = \frac{2}{3}QP = \frac{2a}{3}$$

$$AI = \frac{2}{3}AC$$

Thus assuming $AB = a$, we have $AC = \sqrt{2}a$, $AI = \frac{2\sqrt{2}}{3}a$ and $BI = \frac{2}{3}BE$.

We can apply the Pythagorean theorem to find BE

$$BE = \frac{\sqrt{5}}{2}a \quad \text{this means that} \quad BI = \frac{\sqrt{5}}{3}a$$

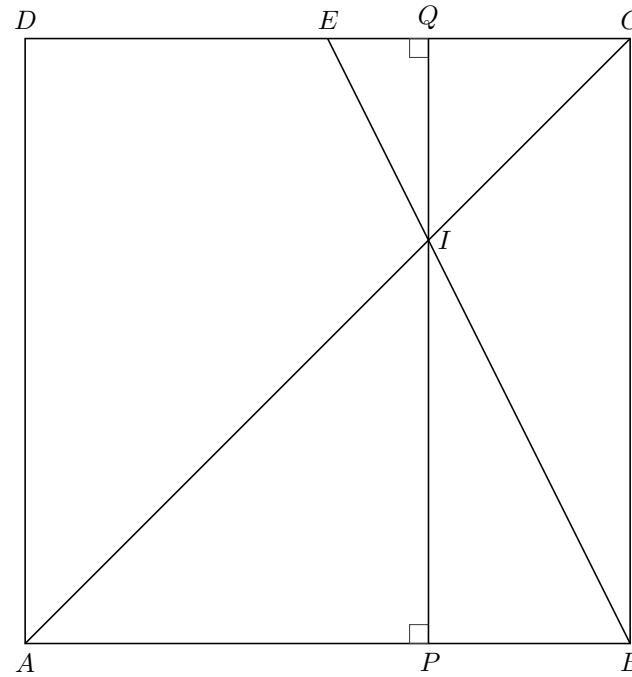
In any triangle, $r \times p = s \times h$, where r is the inradius, p the perimeter, s the side and h the altitude of the triangle.

In other words

$$r \left(1 + \frac{\sqrt{5}}{3} + \frac{2\sqrt{2}}{3} \right) a = a \times \frac{2a}{3}$$

from which r is found:

$$r = \frac{2a}{3 + 2\sqrt{2} + \sqrt{5}}$$

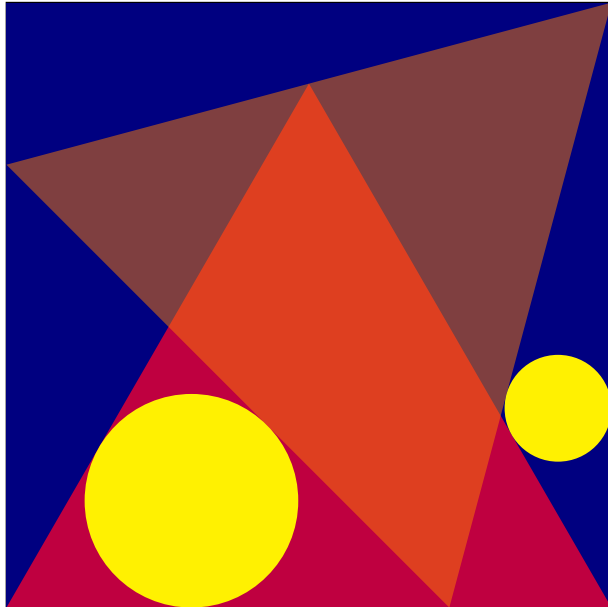


4 Sangaku in a square - Two equilateral triangles

Here is an elegant sangaku that requires both geometric and algebraic skills and some perseverance:

Two equilateral triangles are inscribed into a square as shown in the diagram. Their side lines cut the square into a quadrilateral and a few triangles. Find a relationship between the radii of the two incircles shown in the diagram.

4.1 The figure

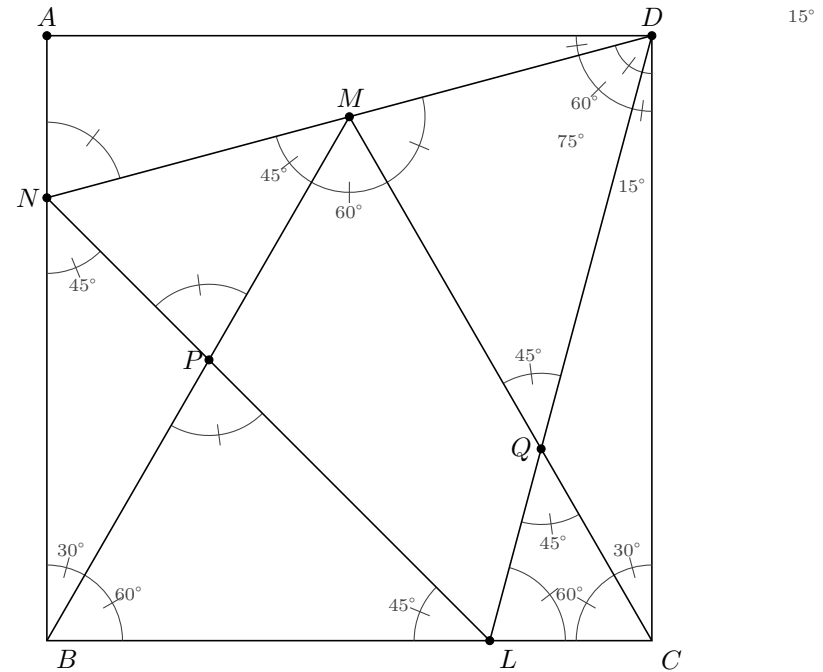


The code of the last figure is

```

\begin{tikzpicture}
  \tkzDefPoint(0,0){B} \tkzDefPoint(8,0){C}%
  \tkzDefPoint(0,8){A} \tkzDefPoint(8,8){D}
  \tkzDrawPolygon(B,C,D,A)
  \tkzClipPolygon(A,B,C,D)
  \tkzFillPolygon[color = blue!50!black](A,B,C,D)
  \tkzDefTriangle[equilateral](B,C) \tkzGetPoint{M}
  \tkzFillPolygon[color = purple](B,C,M)
  \tkzInterLL(D,M)(A,B) \tkzGetPoint{N}
  \tkzDefPointBy[rotation=center N angle -60](D) \tkzGetPoint{L}
  \tkzDefLine[bisector](C,B,M) \tkzGetPoint{x}
  \tkzDefLine[bisector](N,L,B) \tkzGetPoint{y}
  \tkzInterLL(L,y)(B,x) \tkzGetPoint{H}
  \tkzDefLine[bisector](M,C,D) \tkzGetPoint{u}
  \tkzDefLine[bisector](L,D,C) \tkzGetPoint{v}
  \tkzDefPointBy[projection=onto C--B](H) \tkzGetPoint{I}
  \tkzFillPolygon[color = orange,opacity = .5](D,N,L)
  \tkzFillCircle[color = yellow](H,I)
  \tkzInterLL(C,u)(D,v) \tkzGetPoint{K}
  \tkzDefPointBy[projection=onto C--D](K) \tkzGetPoint{J}
  \tkzFillCircle[color = yellow](K,J)
\end{tikzpicture}

```



Firstly, we need to prove that it is possible that two equilateral triangles are inscribed into a square as shown in the diagram. A theorem exists but it is nice to find a solution in this particular case. Let $ABCD$ a square, BCM an equilateral triangle. The line (DM) intersects $[AB]$ at point N . Then we construct a point L on the side $[BC]$ and the angle $\widehat{NDL} = 60^\circ$

The triangle MCD is an isosceles triangle with two sides MC and CD of the same length a . It follows that

$$\widehat{MDC} = \widehat{DMC} = 75^\circ \text{ because } \widehat{MCD} = 30^\circ$$

Now we we can determine the angular size of all the angles

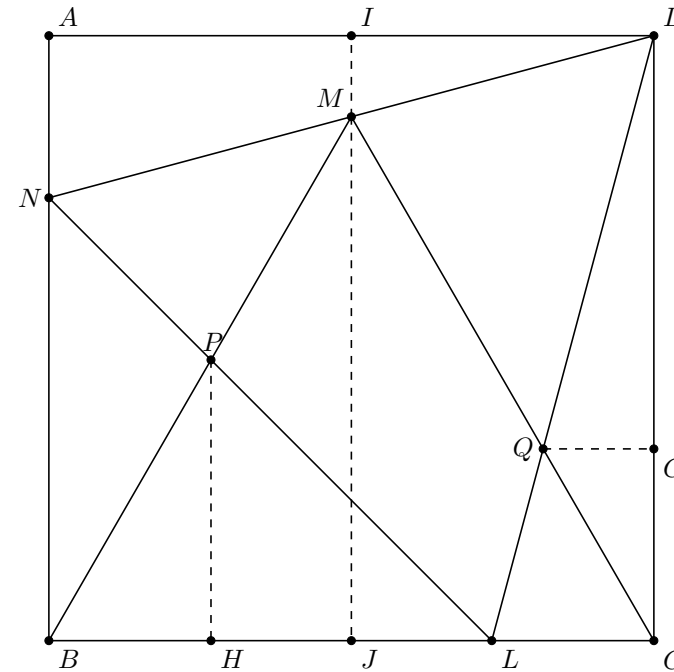
$$\widehat{LDC} = 15^\circ \text{ so } \widehat{ADN} = 15^\circ \text{ and } \widehat{AND} = 75^\circ$$

$$AN = LC \text{ then } BL = BN \text{ and } \widehat{BLN} = \widehat{LNB} = \widehat{NMB} = \widehat{MQD} = \widehat{LQC} = 45^\circ$$

```

\begin{tikzpicture}
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  \tkzDefPoint(0,8){A} \tkzDefPoint(8,8){D}
  \tkzDrawPolygon(B,C,D,A)
  \tkzDefTriangle[equilateral](B,C) \tkzGetPoint{M}
  \tkzInterLL(D,M)(A,B) \tkzGetPoint{N}
  \tkzDefPointBy[rotation=center N angle -60](D) \tkzGetPoint{L}
  \tkzInterLL(N,L)(M,B) \tkzGetPoint{P}
  \tkzInterLL(M,C)(D,L) \tkzGetPoint{Q}
  \tkzMarkAngles[fill = green!50](L,D,C A,D,N)
  \tkzMarkAngles[fill = yellow!50](D,C,M M,B,A)
  \tkzMarkAngles[fill = magenta!50](N,L,B B,N,L N,M,B D,Q,M L,Q,C)
  \tkzMarkAngles[fill = blue!50](D,N,A M,P,N B,P,L C,L,D C,M,D)
  \tkzMarkAngles[fill = red!50](C,B,M B,M,C M,C,B N,D,L)
  \tkzMarkAngle[size = 0.5,fill = blue!50](M,D,C)
  \tkzLabelAngle[pos=2](L,D,C){\scriptsize $15^\circ$}
  \tkzLabelAngle[pos=-2](A,D,N){\scriptsize $15^\circ$}
  \tkzLabelAngles[pos=1.25](D,C,M M,B,A){\scriptsize $30^\circ$}
  \tkzLabelAngles[pos=1.25](N,L,B B,N,L
    N,M,B D,Q,M L,Q,C){\scriptsize $45^\circ$}
  \tkzLabelAngles[pos=1.25](C,B,M B,M,C
    M,C,B N,D,L){\scriptsize $60^\circ$}
  \tkzLabelAngle[pos=1.75](M,D,C){\scriptsize $75^\circ$}
  \tkzDrawSegments(D,N N,L L,D B,M M,C)
  \tkzDrawPoints(L,N,P,Q,M,A,D) \tkzLabelPoints[left](N,P,Q)
  \tkzLabelPoints[above](M,A,D) \tkzLabelPoints(L,B,C)
\end{tikzpicture}

```



We need some preliminaries to find the ratio between the radii of the two incircles shown in the first diagram.

Assume the side of the square equals a , first, we determine MI

$$MJ = \frac{\sqrt{3}}{2}a \text{ and } MI = a - \frac{\sqrt{3}}{2}a = \frac{(2 - \sqrt{3})}{2}a$$

Thus we can find AN and NB

$$AN = 2MI = (2 - \sqrt{3})a$$

and

$$BN = AB - AN = a - (2 - \sqrt{3})a = (\sqrt{3} - 1)a$$

ADN is a right triangle with hypotenuse ND . We have, $AD^2 + AN^2 = ND^2$ by the Pythagorean theorem.

Using this, we continue:

$$ND^2 = a^2 + (2 - \sqrt{3})^2 a^2 = a^2(8 - 4\sqrt{3})$$

$$ND = NL = LD = (\sqrt{6} - \sqrt{2})a$$

We can see that the angles \widehat{DNL} and \widehat{NLD} have the same degree measurements 60° . DNL is an equilateral triangle and it is the largest equilateral triangle which can be inscribed in the square (Madachy 1979). We prove lately that the side is $s = (\sqrt{6} - \sqrt{2})a$.

The value of $\tan(15^\circ)$ which will be useful later on.

$$\tan 15^\circ = \frac{AN}{AD} = \frac{(2-\sqrt{3})a}{a} = 2-\sqrt{3}$$

Now we can apply the standard formula in a triangle to determine the inradius

$$r = \frac{sh}{p}$$

where r, p, s, h are respectively the inradius, perimeter, a side and the altitude to the side in a triangle.

A good idea is to find a relationship between p and h

Let BPL the first triangle, here $h = PH$, $l = BL$ and $p = BP + PL + LB$.

$$\sin(60^\circ) = \frac{\sqrt{3}}{2} = \frac{PH}{BP} = \frac{h}{BP}$$

thus

$$BP = \frac{2h}{\sqrt{3}}$$

HPL is an isosceles right triangle. The hypotenuse PL has length $\sqrt{2}h$.

And finally the relation between BL and h can be obtain like this

$$BL = BH + HL = \frac{h}{\sqrt{3}} + h$$

In an other way

$$BL = BN = (\sqrt{3}-1)a$$

The inradius r_1 of BLP is

$$r_1 = \frac{sh}{p} = \frac{(\sqrt{3}-1)ah}{\frac{2h}{\sqrt{3}} + \sqrt{2}h + \frac{h}{\sqrt{3}} + h} = \frac{(\sqrt{3}-1)a}{1 + \sqrt{2} + \sqrt{3}}$$

For CQD, similarly the inradius r_2 can be found with

$$r_2 = \frac{ah}{p}$$

with $p = CQ + QD + DC$ and $h = QG$

or

$$\frac{QD}{DL} = \frac{QG}{LC} = \frac{h}{(2-\sqrt{3})a}$$

from which

$$QD = \frac{(\sqrt{6}-\sqrt{2})ah}{(2-\sqrt{3})a} = (\sqrt{6}+\sqrt{2})h$$

We continue

$$\frac{QG}{QC} = \frac{1}{2} = \frac{h}{GC}$$

from which

$$QC = 2h$$

and finally

$$DC = DG + GC = \frac{h}{DG} = \tan(15^\circ) = 2 - \sqrt{3}$$

So

$$DG = \frac{h}{2-\sqrt{3}} = (2+\sqrt{3})h$$

and

$$\frac{GC}{QC} = \frac{GC}{2h} = \frac{\sqrt{3}}{2}$$

Thus we can find

$$GC = \sqrt{3}h$$

Finally

$$p = (\sqrt{6}+\sqrt{2})h + 2h + (2+\sqrt{3})h + \sqrt{3}h$$

The second radius is

$$r_2 = \frac{ah}{p} = \frac{ah}{(\sqrt{6}+\sqrt{2})h + 2h + (2+\sqrt{3})h + \sqrt{3}h} = \frac{a}{4 + \sqrt{2} + 2\sqrt{3} + \sqrt{6}}$$

Without a special effort, we conclude that $r_1 = 2r_2$

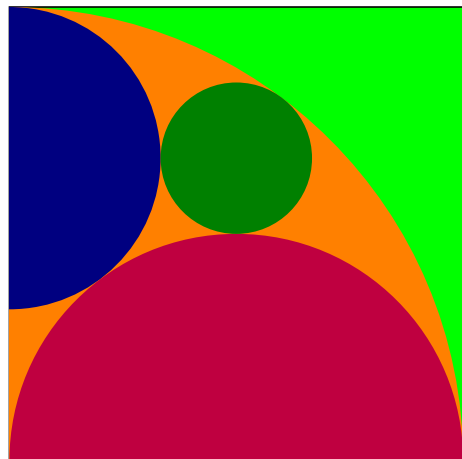
4.3 Construction

There is no problem. First we draw the square ABCD and an equilateral triangle BMC, then we draw the line (CM) . N is the intersection of (CM) with (AB) . We draw the equilateral triangle CNL. P et Q are intersections of sides of the equilateral triangles. Finally we draw the incenters of the triangles BPL and CQD.

5 Sangaku in a square - V

This sangaku requires to determine the relative radii of the circles shown can be solved by an application of the Pythagorean theorem. Find a relationship between the radius of the circle and the side of the square.

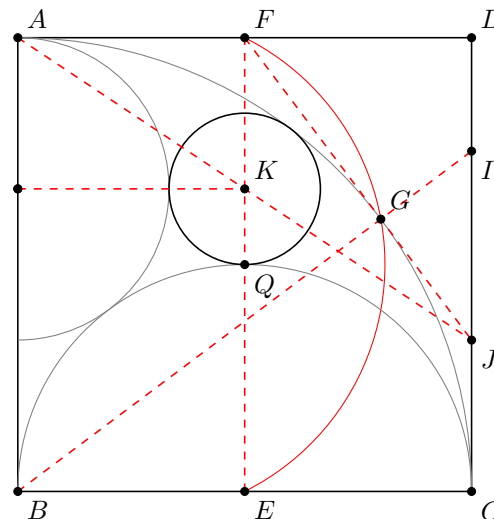
5.1 The picture



```

\begin{tikzpicture}[scale = .75]
  \tkzDefPoint(0,0){B} \tkzDefPoint(8,0){C}%
  \tkzDefPoint(0,8){A} \tkzDefPoint(8,8){D}
  \tkzDrawPolygon[fill=green](A,B,C,D)
  \tkzClipPolygon(A,B,C,D)
  \tkzDefPoint(4,8){F}
  \tkzDefPoint(4,0){E}
  \tkzDefPoint(4,4){Q}
  \tkzTangent[from=B](F,A) \tkzGetPoints{G}{H}
  \tkzInterLL(F,G)(C,D) \tkzGetPoint{J}
  \tkzInterLL(A,J)(F,E) \tkzGetPoint{K}
  \tkzDefPointBy[projection=onto B--A](K) \tkzGetPoint{M}
  \tkzDrawCircle[fill = orange](B,A)
  \tkzDrawCircle[fill = blue!50!black](M,A)
  \tkzDrawCircle[fill = purple](E,B)
  \tkzDrawCircle[fill = yellow](K,Q)
\end{tikzpicture}

```



5.2 Explanation and construction

Assume the radius AM equal r and the side of the square is a .

step 1. Firstly, $ME^2 = BE^2 + BM^2$ and the two circles are tangent if $ME = r + \frac{a}{2}$. The equality becomes

$$\left(r + \frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2 + (a-r)^2$$

$$r = \frac{a}{3}$$

we have

$$ME = BK = \frac{5}{6}a$$

step 2.

$$KQ = KE - QE = \frac{2a}{3} - \frac{a}{2} = \frac{a}{6}$$

and

$$MK - \frac{a}{3} = \frac{a}{2} - \frac{a}{3} = \frac{a}{6}$$

The circle K is tangent mutually at the circle E and the circle M .

step 3. The little circle with center K is tangent at the circle B.

$$a - \frac{5a}{6} = \frac{a}{6}$$

step 4. A good method to obtain K is finally to place I such as $DI = \frac{a}{4}$. Therefore, BI intercepts the big circle in G with $GI = \frac{a}{4}$, $FG = \frac{a}{2}$ and FG orthogonal to BG . FG intercepts BC in J such as $DJ = \frac{2a}{3}$. K is the common point between AJ and EF .

6 Sangaku - Circle Inscribing

References

Weisstein, Eric W. "Circle Inscribing." From MathWorld—A Wolfram Web

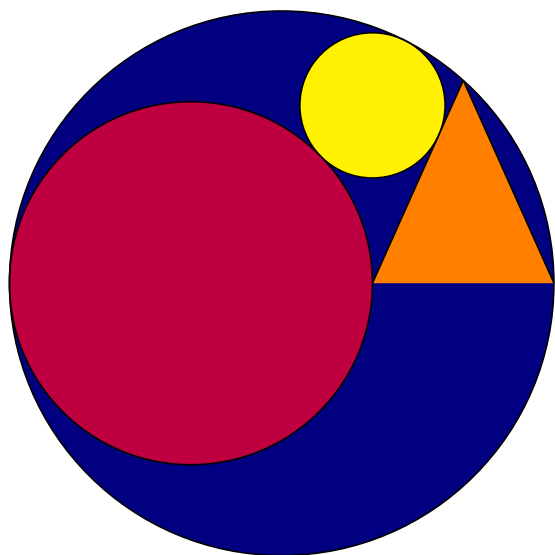
<http://mathworld.wolfram.com/CircleInscribing.html>

Alexander Bogomolny

<http://www.cut-the-knot.org/>

Construct the figure consisting of a circle centered at O , a second smaller circle centered at O_2 tangent to the first, and an isosceles triangle whose base $[AB]$ completes the diameter of the larger circle $[XB]$ through the smaller $[XA]$. Now inscribe a third circle with center O_3 inside the large circle, outside the small one, and on the side of a leg of the triangle. (from an 1803 sangaku found in Gumma Prefecture)

6.1 The figure



6.2 Explanation

To find the explicit position and size of the circle, let the circle with center O have radius R and be centered at O and let the circle with center O_2 have radius r .

$$(r + a)^2 = r^2 + y^2$$

$$(R - a)^2 = (R - 2r)^2 + y^2$$

for a and y gives

$$a = 2r \frac{R - r}{R + r}$$

$$y = AO_3 = 2\sqrt{2Rr} \frac{\sqrt{R - r}}{R + r}$$

but now we need to prove that the circle is tangent to the line AC .

Let α the angle ACD and the angle O_3AC

$$\sin(\alpha) = \frac{AD}{AC}$$

$OD = r$ and $AD = R - r$

OCD is a right triangle with hypotenuse $OC = R$. We have, $OD^2 + CD^2 = OC^2$ by the Pythagorean theorem. In terms of r , the theorem appears as

$$r^2 + CD^2 = R^2$$

which is equivalent to

$$CD^2 = R^2 - r^2$$

and with the right triangle ADC and the Pythagorean theorem

$$AC^2 = AD^2 + CD^2 = (R - r)^2 + R^2 - r^2 = 2R(R - r)$$

finally

$$\sin(\alpha) = \frac{AD}{AC} = \frac{R - r}{\sqrt{2R(R - r)}} = \frac{\sqrt{R - r}}{\sqrt{2R}}$$

Let H the projection point of O_3 on the line AC , and d the length of O_3H

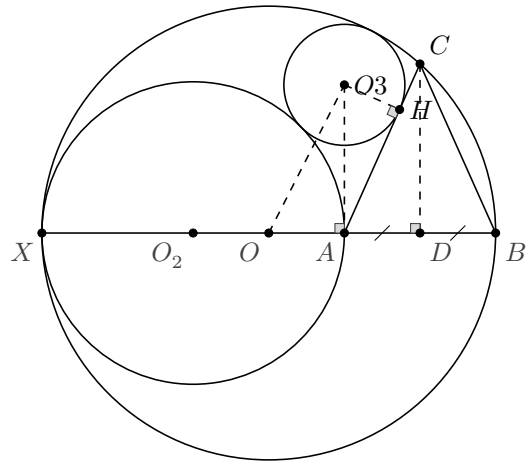
$$\sin(\alpha) = \frac{O_3H}{AO_3} = \frac{d}{y} = \frac{d}{2\sqrt{2Rr} \frac{\sqrt{R - r}}{R + r}}$$

Using the two forms of $\sin(\alpha)$

$$\frac{d}{2\sqrt{2Rr} \frac{\sqrt{R - r}}{R + r}} = \frac{\sqrt{R - r}}{\sqrt{2R}}$$

So

$$d = 2r \frac{R-r}{R+r} = a$$



7 Two Unrelated Circles

Some references

a Scientific American article by Tony Rothman written in co-operation with Hidetoshi Fukagawa; The book by H. Fukagawa and D. Pedoe. <http://www.sangaku.info/> <http://mathworld.wolfram.com> <http://www.wasan.jp/english/> <http://www.cut-the-knot.org/pythagoras/Sangaku.shtml>

Chord $[ST]$ is perpendicular to diameter $[CP]$ of a circle with center O at point R . Q is point of $[CP]$ between P and R . $[SQ]$ intersects the circle in V .

Chords $[KN]$ and $[ST]$ are perpendicular to diameter $[CP]$ of a circle with center O at points Q and R . $[SQ]$ intersects the circle in V . (K, S are on one side of $[CP]$, N and T on the other. Q is between P and R .) Let r be the radius of the circle inscribed into the curvilinear triangle TQV . Prove that

$$\frac{1}{r} = \frac{1}{PQ} + \frac{1}{QR}$$

7.1 The figure

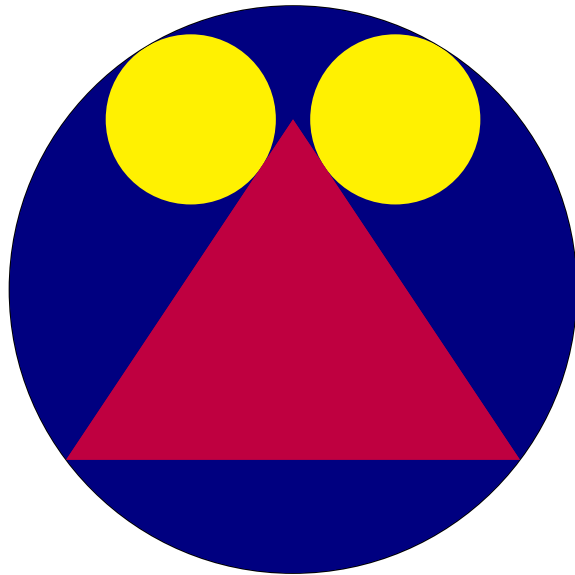


Figure 1: Sangaku Two Unrelated Circles

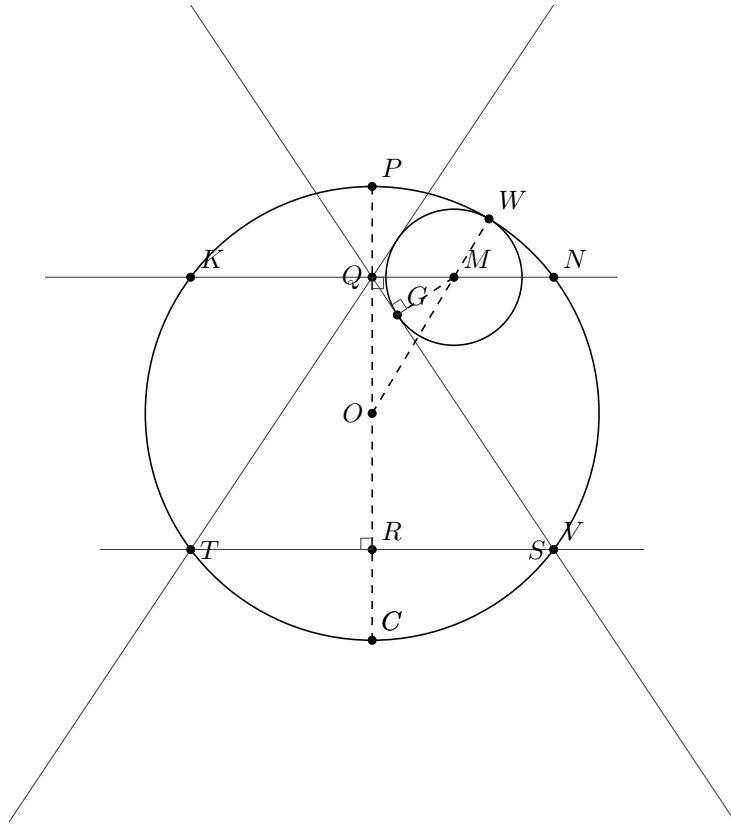
Now you can read the code to get the last picture.

```
\begin{tikzpicture}[scale=.75]
  \tkzDefPoint(0,0){O} \tkzDefPoint(-2,-3){A}
  \tkzDefPoint(2,-3){B} \tkzDefPoint(0,3){Q}
  \tkzDrawCircle[R](O,5 cm)
  \tkzInterLC[R](A,B)(O,5 cm)
  \tkzGetPoints{S}{T}
  \tkzInterLC[R](S,Q)(O,5 cm)
  \tkzGetSecondPoint{X}
  \tkzInterLC[R](T,Q)(O,5 cm)
  \tkzGetFirstPoint{Z}
  \tkzDefMidPoint(S,T) \tkzGetPoint{R}
  \tkzDefLine[orthogonal=through Q](O,Q)
  \tkzCalcLength(R,Q)\tkzGetLength{dRQ}
  \tkzCalcLength(S,Q)\tkzGetLength{dSQ}
  \tkzDefPoint(\dSQ/\dRQ*1.5,3){M}
  \tkzDefPoint(-\dSQ/\dRQ*1.5,3){N}
  \tkzDefPointBy[projection=onto S--Q](M)
  \tkzGetPoint{G}
  \tkzInterLC[R](O,M)(O,5 cm)
  \tkzGetPoints{W}{Y}
  \tkzClipCircle[R](O,5 cm)
  \tkzDrawLines[add= 1 and 1](A,B S,Q T,Q)
  \tkzDrawCircle[R](M,1.5 cm)
  \tkzFillCircle[R,color = blue!50!Black](O,5cm)
  \tkzFillCircle[R,color = yellow](M,1.5cm)
  \tkzFillCircle[R,color = yellow](N,1.5cm)
  \tkzFillPolygon[color = purple](S,T,Q)
  \tkzFillPolygon[color = purple](Z,X,Q)
\end{tikzpicture}
```

7.2 Explanation : Idea of Nathan Bowler

Take coordinates such that it is the unit circle ($r = 1$) with origin M and with Q on the x axis (Q is between O and P). Let $G = (a ; b)$, $W = (u ; v)$, $M = (0 ; 0)$ with $a < 0$ and $u > 0$

Then:



- The line (QS) has equation $ax + by = 1$, $(MG = 1$ and (MG) perpendicular to (QS))
- The line (MW) has equation $xv - yu = 0$,
- Q is at $\left(\frac{1}{a}; 0\right)$,
- O is at $\left(\frac{1}{a}; \frac{v}{au}\right)$,
- $OQ = -\frac{v}{au}$,

$$\begin{aligned} OW^2 &= \left(1 - \frac{1}{au}\right)^2 \\ &= \left(u - \frac{1}{a}\right)^2 + \left(v - \frac{v}{au}\right)^2 \\ &= u^2 - \frac{2u}{a} + \frac{1}{a^2} + v^2 - \frac{v^2}{u^2 a^2} - \frac{2v^2}{ua} \end{aligned}$$

- $OW^2 = 1 - \frac{2u}{a} + \frac{1}{a^2} - \frac{1-u^2}{u^2 a^2} - \frac{2-2u^2}{ua} = \left(1 - \frac{1}{ua}\right)^2$.

- S and V are points of circles with center O and M , so if $(x; y)$ are coordinates of these points

$$u^2 + v^2 = 1 \text{ and } \left(u - \frac{1}{a}\right)^2 + \left(v - \frac{v}{au}\right)^2 = \left(1 - \frac{1}{ua}\right)^2$$

Further, we obtain

$$y^2 - \frac{av}{u} 2y + 2\frac{a}{u} - a^2 - 1 = 0$$

The discriminant is $\left(\frac{a^2}{u^2} - 2\frac{a}{u} + 1\right)^2 = \left(\frac{a}{u} - 1\right)^2 = \left(1 - \frac{a}{u}\right)^2$

This equation has two solutions :

$$y_1 = \frac{av}{u} - \left(1 - \frac{a}{u}\right) \text{ negative and } y_2 = \frac{av}{u} + \left(1 - \frac{a}{u}\right) \text{ positive.}$$

Finally

$$\begin{aligned} QR &= -y_1 \\ &= 1 - \frac{a}{u} - \frac{av}{u}. \end{aligned}$$

and

$$PQ = OP - OQ = OW - OQ = 1 - \frac{1}{ua} + \frac{v}{au}$$

$$PQ = 1 - \frac{1}{au} + \frac{v}{au} = 1 - \frac{1-v}{au}.$$

So

$$(PQ - 1)(QR - 1) = \left(-\frac{a}{u} - \frac{av}{u}\right) \left(-\frac{1-v}{au}\right) = \frac{(1-v)(1+v)}{u^2} = 1.$$

Equivalently,

$$\frac{1}{PQ} + \frac{1}{QR} = 1 = \frac{1}{r}.$$

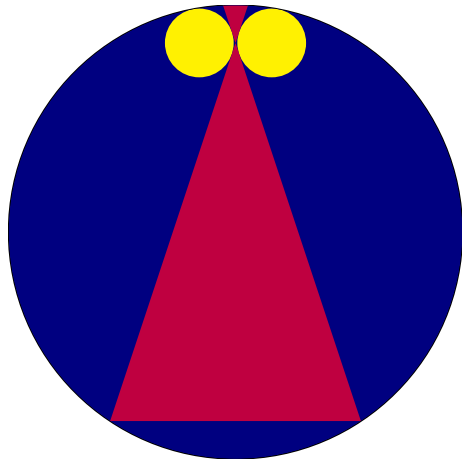


Figure 2: Sangaku-Two Unrelated Circles : $R = 6$ cm, $OR = -5$ cm et $RQ = 10$ cm

```
\begin{tikzpicture}[scale = .5]
  \TwoUnrelatedCircles{6}{-5}{10}
\end{tikzpicture}
```

In the general case, some information will be needed. For instance, it is necessary to give the radius of the big circle, the position of points R and Q . I have decided to give $R(0, r)$ relatively to O and Q relatively to P with the value of PQ .

7.3 A macro `\TwoUnrelatedCircles`

The macro, below, is used to obtain some examples.

```
\newcommand*{\TwoUnrelatedCircles}[3]{
  % #1 --> R ; #2 --> Y_P <0 or >0 ; #3 --> PQ >= 0
  \xdef\ORadius{#1} \xdef\tucR{#2} \xdef\tucQR{#3}
  \pgfmathparse{\tucQR+\tucR} \xdef\tucQ{\pgfmathresult}
  \pgfmathparse{\ORadius-(\tucQR+\tucR)}
  \xdef\tucPQ{\pgfmathresult}
  \pgfmathparse{(\tucPQ*\tucQR)/(\tucPQ+\tucQR)}
  \let\tucr\pgfmathresult
  \pgfmathparse{\ORadius-\tucr}
  \let\OORadius\pgfmathresult%
  \tkzDefPoint(0,0){O}
  \tkzDefPoint(-\ORadius, \tucR){A}
  \tkzDefPoint(\ORadius, \tucR){B}
  \tkzDefPoint(0, \tucQ){Q}
  \tkzInterLC[R](A,B)(O, \ORadius cm)\tkzGetPoints{S}{T}
  \tkzInterLC[R](S,Q)(O, \ORadius cm)\tkzGetPoints{V}
  \tkzInterLC[R](Q,T)(O, \ORadius cm)\tkzGetPoints{Y}{T}
  \tkzDefPoint(0, \tucR){R}
  \tkzDefPoint(-\ORadius, \tucQ){U}
  \tkzInterLC[R](U,Q)(O, \OORadius cm)\tkzGetPoints{M}{N}
  \tkzDrawCircle[R](O, \ORadius cm)
  \tkzClipCircle[R](O, \ORadius cm)
  \tkzDrawLines[add= 1 and 1](S,Q T,Q)
  \tkzDrawCircle[R](M, \tucr cm)
  \tkzFillCircle[R,color = blue!50!Black](O, \ORadius cm)
  \tkzFillCircle[R,color = yellow](M, \tucr cm)
  \tkzFillCircle[R,color = yellow](N, \tucr cm)
  \tkzFillPolygon[color = purple](S,T,Q)
  \tkzFillPolygon[color = purple](V,Y,Q)
}
```

8 Sangaku - Harmonic mean

8.1 Harmonic mean of two numbers

$$\frac{2}{c} = \frac{1}{a} + \frac{1}{b}$$

which says that c is the harmonic mean of a and b .

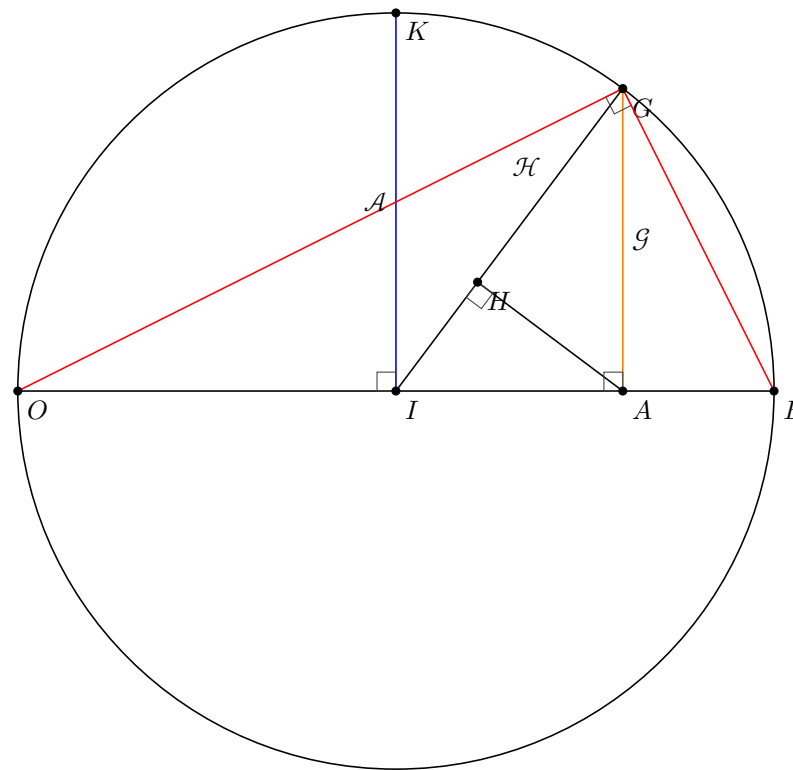
For two numbers a and b , $\mathcal{A} = \frac{a+b}{2}$ and $\mathcal{G} = \sqrt{ab}$, and \mathcal{H} such as $\frac{2}{\mathcal{H}} = \frac{1}{a} + \frac{1}{b}$

- a and b two numbers such as $OA = a$ and $AB = b$. I is the center of the circle \mathcal{C} with diameter $[OB]$. $[IK]$ is a radius perpendicular to (OB) . It's easy to prove $IK = \mathcal{A}$.

- (AG) is a line perpendicular to (OB) in A and G is point of \mathcal{C} . OGB is a right triangle, so

$$AG^2 = OA \times OB$$

Finally we get $AG = \mathcal{G} = \sqrt{ab}$



GAH and IAG are right rectangles and these rectangles are similar so :

$$\frac{GH}{AG} = \frac{AG}{IH} \text{ and } \frac{AG}{\mathcal{G}} = \frac{\mathcal{G}}{\mathcal{A}}$$

In this case, we have

$$\mathcal{G}^2 = \mathcal{A} \times AG \text{ and } AG = \frac{\mathcal{G}^2}{\mathcal{A}}$$

Now we can prove $AG = \mathcal{H}$

$$\frac{1}{\mathcal{H}} = \frac{\mathcal{A}}{\mathcal{G}^2} = \frac{a+b}{2ab}$$

Finally

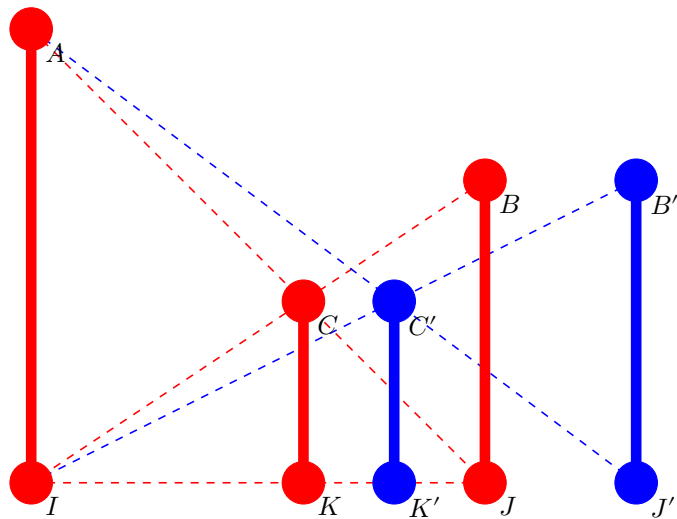
$$\frac{1}{2} \left(\frac{1}{a} + \frac{1}{b} \right) = \frac{1}{\mathcal{H}}$$

8.2 Sangaku - Harmonic mean

Two vertical segments AI, BJ ($AI = a$ and $BJ = b$), the intersection C of the diagonals is at the height that depends solely on AI and BJ . In fact

$$\frac{1}{CK} = \frac{1}{AI} + \frac{1}{BJ} = \frac{1}{a} + \frac{1}{b}$$

now if $B'J' = BJ$ then $C'K' = CK$.



```
\begin{tikzpicture}
\tkzDefPoint(0,6){A}\tkzDefPoint(0,0){I}
\tkzDefPoint(6,4){B}\tkzDefPoint(8,4){B'}
\tkzDefPoint(6,0){J}\tkzDefPoint(8,0){J'}
\tkzInterLL(A,J)(B,I)\tkzGetPoint{C}
\tkzDefPointBy[projection=onto I--J](C)\tkzGetPoint{K}
\tkzInterLL(A,J')(B',I)\tkzGetPoint{C'}
\tkzDefPointBy[projection=onto I--J](C')\tkzGetPoint{K'}
\tkzDrawSegments[color = red,style=dashed](A,J B,I I,J)
\tkzDrawSegments[color = blue,style=dashed](A,J' B',I)
\tkzDrawSegments[line width = 4pt,color = red](C,K A,I B,J)
\tkzDrawSegments[line width = 4pt,color = blue](C',K' B',J')
\tkzDrawPoints[color = red,size = 16](A,I,C,K,B,J)
\tkzDrawPoints[color = blue,size = 16](C',K',B',J')
\tkzLabelPoints(A,I,C,K,B,J,C',K',B',J')
\end{tikzpicture}
```

Let's denote $AI = a, BJ = b, CK = c, IK = \alpha$ and $KJ = \beta$.
 The triangles AIJ and CKJ are similar as are triangles BJI and CKI .
 We have the proportion

$$\frac{c}{a} = \frac{\beta}{\alpha + \beta} \text{ and } \frac{c}{b} = \frac{\alpha}{\alpha + \beta}$$

We can add the two equalities

$$\frac{c}{a} + \frac{c}{b} = \frac{\beta}{\alpha + \beta} + \frac{\alpha}{\alpha + \beta} = 1$$

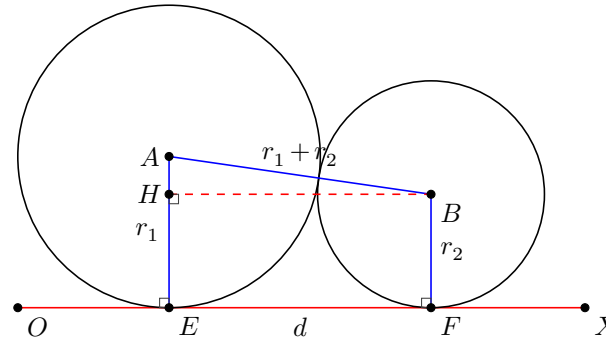
A division by c gives the desired result:

$$\frac{1}{c} = \frac{1}{a} + \frac{1}{b}$$

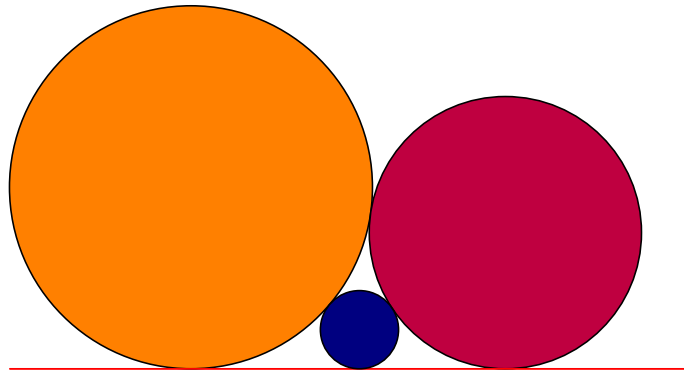
which says that c is the half of the harmonic mean of a and b .

9 Sangaku - Three Tangent Circles

Given three circles tangent to each other and to a straight line, express the radius of the middle circle via the radii of the other two. This problem was given as a Japanese temple problem on a tablet from 1824 in the Gumma Prefecture (MathWorld)



9.1 The figure



```
\begin{tikzpicture}[scale=.6]
  \tkzDefPoint(4,4){A}\tkzDefPoint(10.928,3){B}
  \tkzDefPoint(0,0){O}\tkzDefPoint(15,0){X}
  \tkzDrawSegment[color = red](O,X)
  \tkzDrawCircle[R,fill = orange](A,4 cm)
  \tkzDrawCircle[R,fill = purple](B,3 cm)
  \pgfmathparse{4+8*sqrt(3)/(2+sqrt(3))}
  \edef\x{\pgfmathresult}
  \pgfmathparse{12/((2+sqrt(3))*(2+sqrt(3)))}
  \edef\y{\pgfmathresult}
  \tkzDefPoint(\x,\y){C}
  \tkzDrawSegment[color = red](O,X)
  \tkzDrawCircle[R,fill = blue!50!black](C,\y cm)
\end{tikzpicture}
```

9.2 Explanation

Lemma : Given two circles tangent to each other and to a straight line at points E and F, we can express EF with the radii (r_1, r_2) of the two circles :

$$EF = 2\sqrt{r_1 r_2}$$

Proof : ABH is a right triangle with hypotenuse $AB = r_1 + r_2$. We have by the Pythagorean theorem

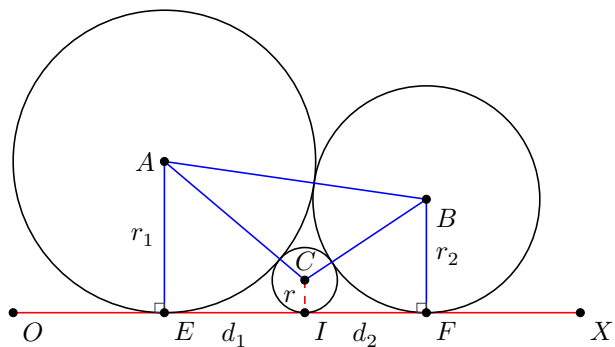
$$AB^2 = AH^2 + HB^2 \\ = AH^2 + d^2$$

If $r_1 > r_2$

$$AB^2 = AH^2 + HB^2 \\ (r_1 + r_2)^2 = (r_1 - r_2)^2 + d^2$$

Finally $d^2 = 4r_1 r_2$ and $d = 2\sqrt{r_1 r_2}$.

Now we have two circles ($r_1 > r_2$) tangent to each other and to a straight line, and we want to draw another circle tangent to each others and to the same straight line :



We can apply the lemma to the circles (A, r_1) and (B, r_2) , then to the circles (A, r_1) and (C, r) and finally to the circles (B, r_2) and (C, r) .

We get four equations :

$$\begin{aligned} d &= d_1 + d_2 \\ d &= 2\sqrt{r_1 r_2} \\ d_1 &= 2\sqrt{r_1 r} \\ d_2 &= 2\sqrt{r r_2} \end{aligned}$$

After simplification, the equations become

$$2\sqrt{r_1 r_2} = 2\sqrt{r_1 r} + 2\sqrt{r r_2}$$

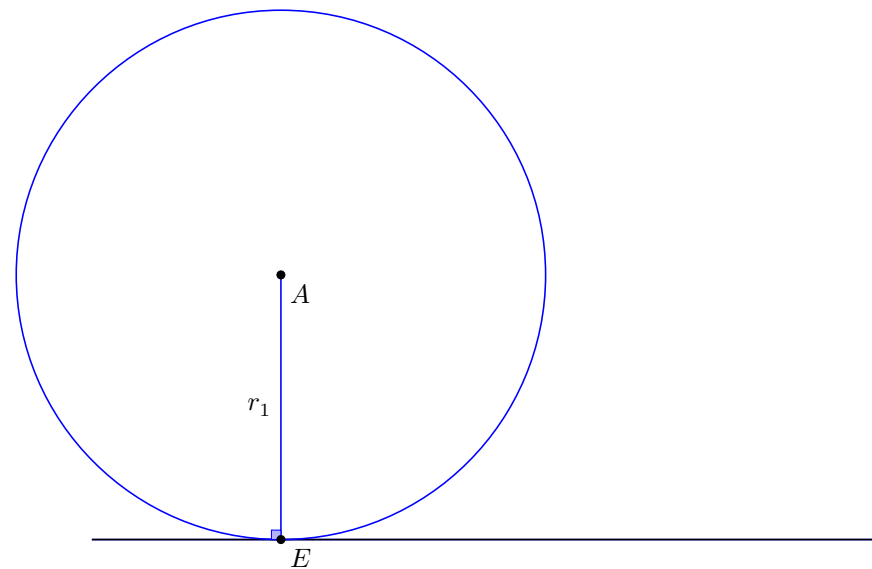
Divide now by $\sqrt{r}\sqrt{r_1}\sqrt{r_2}$ to obtain

$$\frac{1}{\sqrt{r}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$$

This is the condition between the radii.

9.3 How to make this construction with a ruler and a compass

step 1. We draw a line (OX) and a circle with center A tangent to the line in E. The radius of this circle is $EA = r_1 = 7$ cm.



step 2. Now we want to draw a second circle with center B of radius $r_2 = FB = 4$ cm mutually tangent to the first circle and to the line (OX) in F.

We place a point P on the line (EA) such as $AP = r_2 = 4$ cm

The circle with diameter AP intercepts the (OX) axis in a point D such as the length $OD = \sqrt{r_1 r_2}$.

