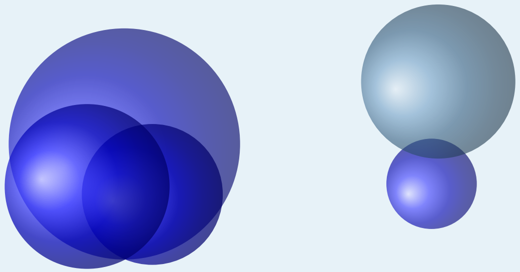
Sangakus with Lua 1.00c



Alain Matthes

February 23, 2026 Documentation V.2.00

http://altermundus.fr

Sangakus with Lua

AlterMundus

Alain Matthes

This document is a little gallery of some simple sangakus. I made all these pictures with my package **`tkz-elements V5.02`** and **`tkz-euclide.sty V5.13`** (option: mini). Sangaku or San Gaku are colorful wooden tablets which were hung often in shinto shrines and sometimes in buddhist temples in Japan and posing typical and elegant mathematical problems. The problems featured on the sangaku are problems of japanese mathematics (wasan). The earliest sangaku found date back to the beginning of the 17th century.

☞ I would like to thank `Till Tantau` for the beautiful LATEX package, namely TikZ.

☞ You will find most of the examples in this document and many other examples on my website : altermundus.fr

Contents

List of Figures

1 Sangaku - References

– ”Japanese Temple Geometry.” a Scientific American article by Tony Rothman written in co-operation with Hidetoshi Fukagawa (Sci. Amer. 278, 85-91, May 1998) ;

– Japanese Temple Geometry Problems the book by H. Fukagawa and D. Pedoe (1989);

– http://www.sangaku.info/;

– http://mathworld.wolfram.com;

– http://www.wasan.jp/english/;

– http://www.cut-the-knot.org/pythagoras/Sangaku.shtml.

2 Sangaku in a square - A very simple sangaku.

2.1 The picture

Find a relationship between the radius of the yellow circle and the side of the square.

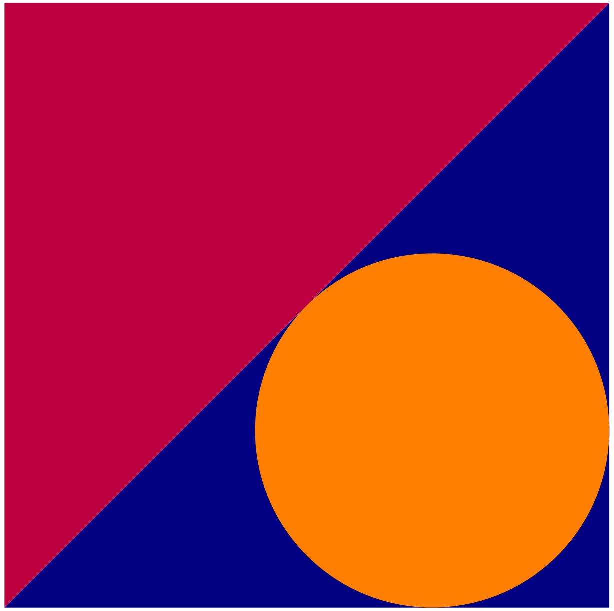


Figure 1: Sangaku 3b in a square

2.2 Explanation

Firstly, we can found the relationship between the inradius and the sides of a right triangle.
If r is the inradius of a circle inscribed in a right triangle with sides a and b and hypotenuse c, then

$$r = \frac{1}{2}(a+b-c).$$

Let ABC represents a right triangle, with the right angle located at C, as shown on the figure. Let a, b and c the lengths of the three sides; c is the length of the hypothenuse.

Let r and p be the radius of the incercle and the semiperimeter of the triangle. a, b and c can be regarded in relation to r and they may be expressed with r: $a = r + (a-r)$, $b = r + (b-r)$ and $c = (a-r) + (b-r)$.
In a right triangle, we have the relation $r = p/2 - c$. From the diagram, the hypotenuse AB is split in two pieces: $(a-r)$ and $(b-r)$, the length of the hypothenuse is $c = (a-r) + (b-r)$.
The perimeter is a function of r

$$p = a+b+c = r+(a-r)+r+(b-r)+(a-r)+(b-r) = 2a+2b-2r$$

so we can expressed r with s and c

$$2r = a+b-c = p-2c \text{ and } r = \frac{p}{2} - c = \frac{a+b-c}{2}$$

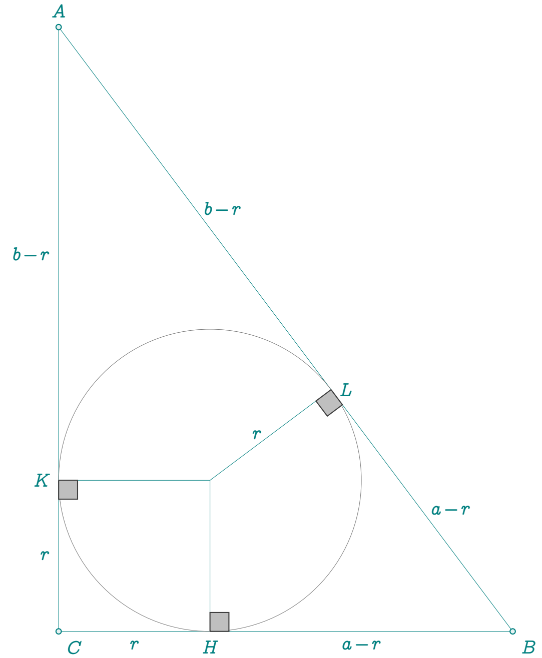


Figure 2: Sangaku 3a

Now, let ABC represents a isosceles right triangle with $AB = AC = a$, then $AC = \sqrt{2}a$ and $a+b-c = 2a-\sqrt{2}a$
So the inradius in this case is

$$r = \frac{2-\sqrt{2}}{2}a$$

Now we can obtain the incenter without the bisectors

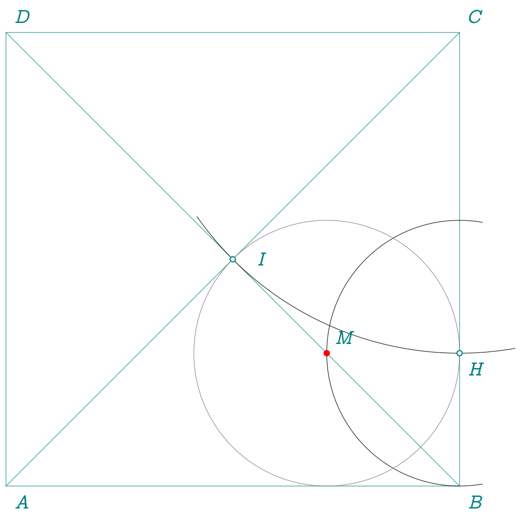


Figure 3: Sangaku 3c

3 Sangaku in a square - Circle and semicircle

Find a relationship between the radius of the yellow circle and the side of the square.

3.1 The picture

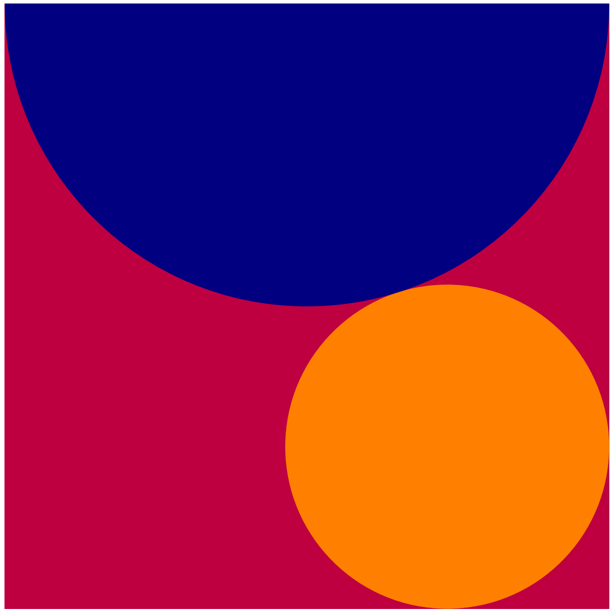


Figure 4: Sangaku 5a in a square

3.2 Explanation

FNE is a right triangle with hypotenuse [EF]. We have, $EN^2 + NF^2 = EF^2$ by the Pythagorean theorem. In terms of a and r, the theorem appears as

$$\left(\frac{a}{2} - r\right)^2 + (a - r)^2 = \left(\frac{a}{2} + r\right)^2$$

which is equivalent to

$$r^2 - 4ar + a^2 = r^2 - 4ar + 4a^2 - 3a^2 = (r - 2a - \sqrt{3}a)(r - 2a + \sqrt{3}a) = 0$$

And finally

$$r = a(2 - \sqrt{3}) < a$$

3.3 Construction

It's easy to prove : $BJ = a(1 - \frac{\sqrt{3}}{2})$ and $BL = a(2 - \sqrt{3})$. First we need to find the point I, then J. K is the intersection of (IJ) and the bisector line of the angle $\widehat{ABC}$. Finally, F is the symmetric point of B around K.

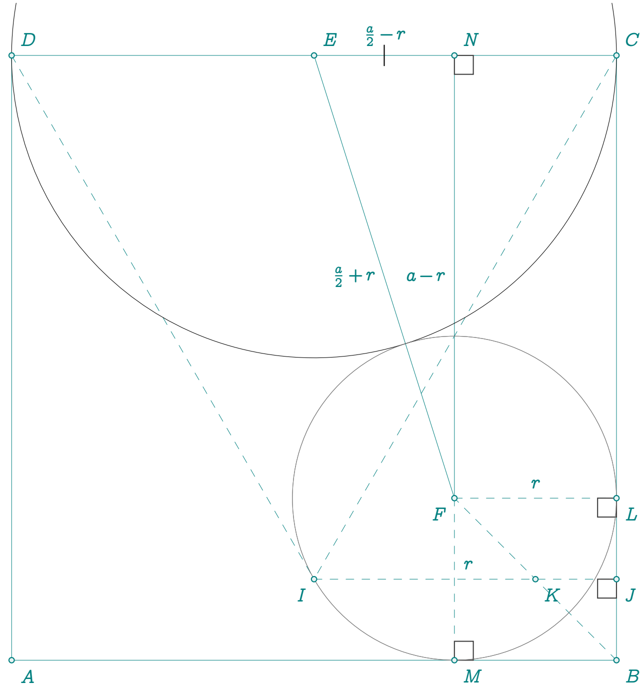


Figure 5: Sangaku 5b

4 Sangaku in a square - two inscribed circles

In the following diagram, a triangle is formed by a line that joins the base of a square with the midpoint of the opposite side and a diagonal. Find the radius of the two inscribed circles.

4.1 The picture

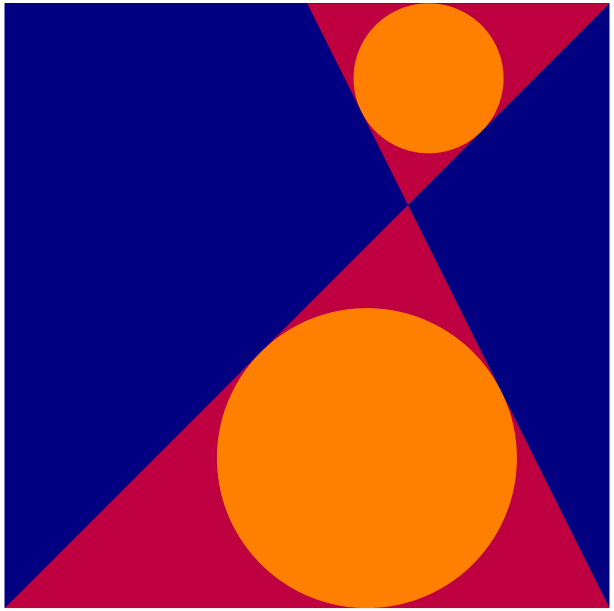


Figure 6: Sangaku 4 in a square - two inscribed circles

4.2 Explanation and construction

QC is parallel to the base AB and is half as long which implies that the two triangles QIC and QAB are similar. I divides the segments QP and AD in ratio 2:1 so that

$$IP = \frac{2}{3}QP = \frac{2a}{3}$$

$$AI = \frac{2}{3}AC$$

Thus assuming $AB = a$, we have $AC = \sqrt{2}a$, $AI = \frac{2\sqrt{2}}{3}a$ and $BI = \frac{2}{3}BE$.
We can apply the Pythagorean theorem to find BE

$$BE = \frac{\sqrt{5}}{2}a \quad \text{this means that} \quad BI = \frac{\sqrt{5}}{3}a$$

In any triangle, $r \times p = s \times h$, where r is the inradius, p the perimeter, s the side and h the altitude of the triangle.
In other words

$$r\left(1+\frac{\sqrt{5}}{3}+\frac{2\sqrt{2}}{3}\right)a = a \times \frac{2a}{3}$$

from which r is found:

$$r = \frac{2a}{3+2\sqrt{2}+\sqrt{5}}$$

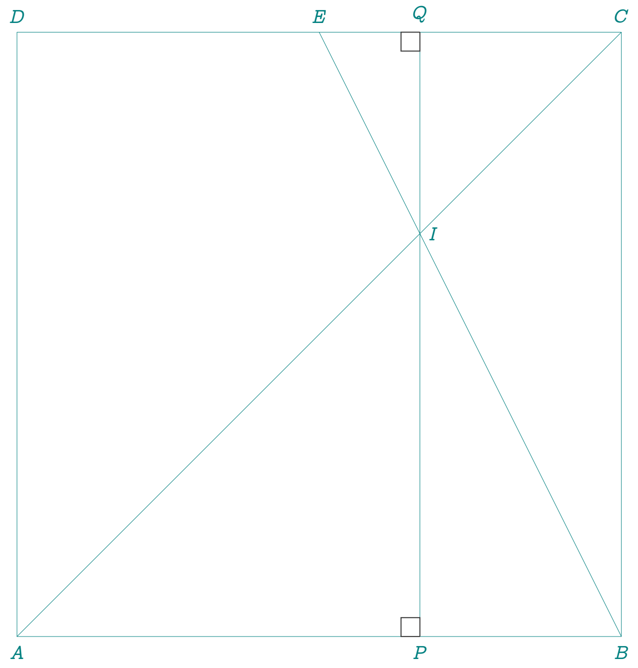


Figure 7: Sangaku 4b

5 Sangaku in a square - Two equilateral triangles

Here is an elegant sangaku that requires both geometric and algebraic skills and some perseverance:
Two equilateral triangles are inscribed into a square as shown in the diagram. Their side lines cut the square into a quadrilateral and a few triangles. Find a relationship between the radii of the two incircles shown in the diagram.

5.1 The figure

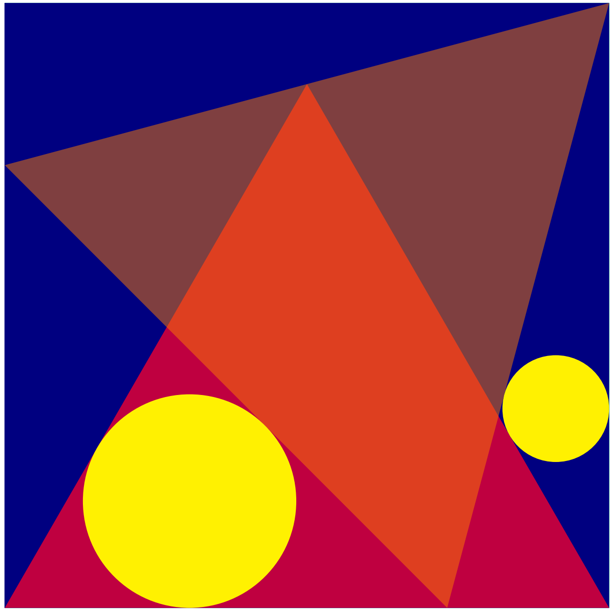


Figure 8: Sangaku 2a in a square - Two equilateral triangles

Firstly, we need to prove that it is possible that two equilateral triangles are inscribed into a square as shown in the diagram. A theorem exists but it is nice to find a solution in this particuler case. Let $ABCD$ a square, BCM an equilateral triangle. The line (DM) intersects $[AB]$ at point N. Then we construct a point L on the side $[BC]$ and the angle $\widehat{NDL} = 60°$
The triangle MCD is an isosceles triangle with two sides MC and CD of the same length a. It follows that

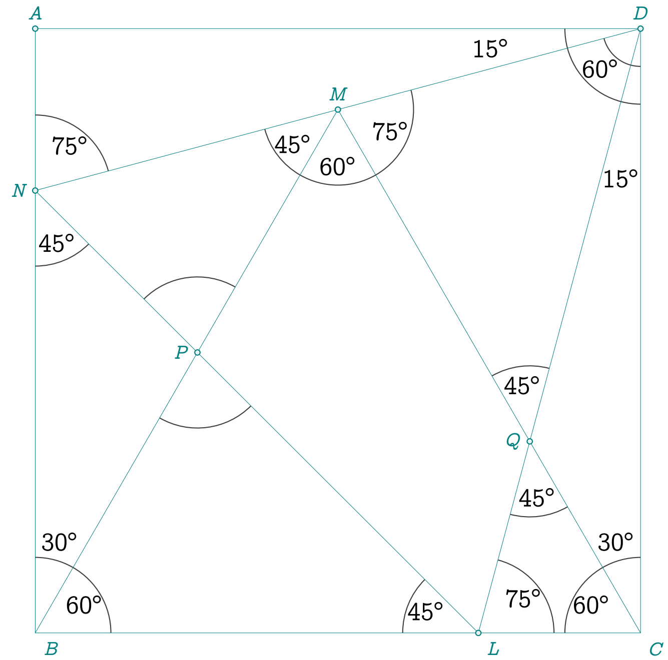


Figure 9: Sangaku 2b

$\widehat{MDC} = \widehat{DMC} = 75°$ because $\widehat{MCD} = 30°$
Now we we can determine the angular size of all the angles
$\widehat{LDC} = 15°$ so $\widehat{ADN} = 15°$ and $\widehat{AND} = 75°$
$AN = LC$ then $BL = BN$ and $\widehat{BLN} = \widehat{LNB} = \widehat{NMB} = \widehat{MQD} = \widehat{LQC} = 45°$

5.2 Explanation

We can see that the angles $\widehat{DNL}$ and $\widehat{NLD}$ have the same degree measurements 60°. DNL is an equilateral triangle and it is the largest equilateral triangle which can be inscribed in the square (Madachy 1979). We prove lately that the side is $s = (\sqrt{6} - \sqrt{2})a$.
We need some preliminaries to find the ratio between the radii of the two incircles shown in the first diagram.

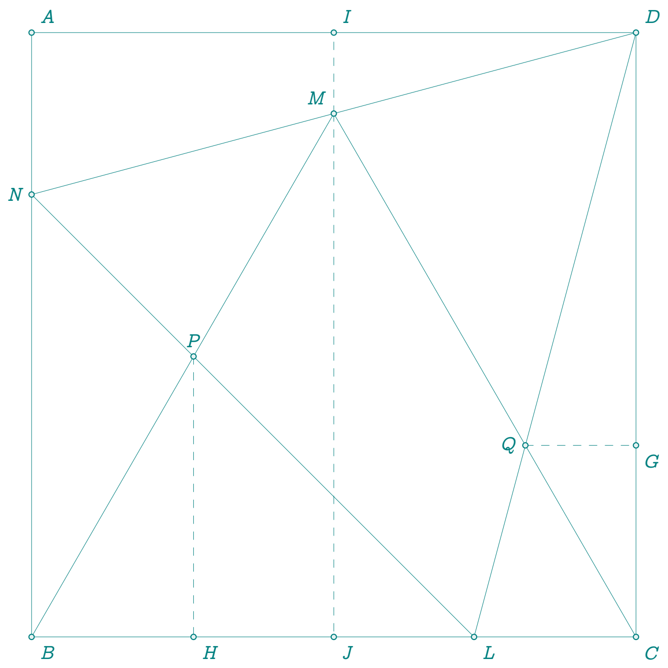


Figure 10: Sangaku 2c

Assume the side of the square equals a, first, we determine MI

$$MJ = \frac{\sqrt{3}}{2}a \text{ and } MI = a - \frac{\sqrt{3}}{2}a = \frac{(2-\sqrt{3})}{2}a$$

Thus we can find AN and NB

$$AN = 2MI = (2-\sqrt{3})a$$

and

$$BN = AB - AN = a - (2-\sqrt{3})a = (\sqrt{3}-1)a$$

ADN is a right triangle with hypotenuse ND. We have, $AD^2 + AN^2 = ND^2$ by the Pythagorean theorem.
Using this, we continue:

$$ND^2 = a^2 + (2-\sqrt{3})^2 a^2 = a^2(8-4\sqrt{3})$$

$$ND = NL = LD = (\sqrt{6}-\sqrt{2})a$$

The value of $\tan(15°)$ which will be useful later on.

$$\tan 15° = \frac{AN}{AD} = \frac{(2-\sqrt{3})a}{a} = 2-\sqrt{3}$$

Now we can apply the standard formula in a triangle to determine the inradius

$$r = \frac{sh}{p}$$

where r, p, s, h are respectively the inradius, perimeter, a side and the altitude to the side in a triangle.
A good idea is to find a relationship between p and h
Let BPL the first triangle, here $h = PH$, $l = BL$ and $p = BP + PL + LB$.

$$\sin(60°) = \frac{\sqrt{3}}{2} = \frac{PH}{BP} = \frac{h}{BP}$$

thus

$$BP = \frac{2h}{\sqrt{3}}$$

HPL is an isosceles right triangle. The hypotenuse PL has length $\sqrt{2}h$.
And finally the relation between BL and h can be obtain like this

$$BL = BH + HL = \frac{h}{\sqrt{3}} + h$$

In an other way

$$BL = BN = (\sqrt{3}-1)a$$

The inradius r_1 of BLP is

$$r_1 = \frac{sh}{p} = \frac{(\sqrt{3}-1)ah}{\frac{2h}{\sqrt{3}} + \sqrt{2}h + \frac{h}{\sqrt{3}} + h} = \frac{(\sqrt{3}-1)a}{1+\sqrt{2}+\sqrt{3}}$$

For CQD, similarly the inradius r_2 can be found with

$$r_2 = \frac{ah}{p}$$

with $p = CQ + QD + DC$ and $h = QG$
or

$$\frac{QD}{DL} = \frac{QG}{LC} = \frac{h}{(2-\sqrt{3})a}$$

from which

$$QD = \frac{(\sqrt{6}-\sqrt{2})ah}{(2-\sqrt{3})a} = (\sqrt{6}+\sqrt{2})h$$

We continue

$$\frac{QG}{QC} = \frac{1}{2} = \frac{h}{GC}$$

from which

$$QC = 2h$$

and finally

$$DC = DG + GC = \frac{h}{DG} = \tan(15°) = 2-\sqrt{3}$$

So

$$DG = \frac{h}{2-\sqrt{3}} = (2+\sqrt{3})h$$

and

$$\frac{GC}{QC} = \frac{GC}{2h} = \frac{\sqrt{3}}{2}$$

Thus we can find

$$GC = \sqrt{3}h$$

Finally

$$p = (\sqrt{6}+\sqrt{2})h + 2h + (2+\sqrt{3})h + \sqrt{3}h$$

The second radius is

$$r_2 = \frac{ah}{p} = \frac{ah}{(\sqrt{6}+\sqrt{2})h + 2h + (2+\sqrt{3})h + \sqrt{3}h} = \frac{a}{4+\sqrt{2}+2\sqrt{3}+\sqrt{6}}$$

Without a special effort, we conclude that $r_1 = 2r_2$

5.3 Construction

There is no problem. First we draw the square ABCD and an equilateral triangle BMC, then we draw the line (CM). N is the intersection of (CM) with (AB). We draw the equilateral triangle CNL. P et Q are intersections of sides of the equilateral triangles. Finally we draw the incenters of the triangles BPL and CQD.

6 Sangaku in a square - 3 Arcs and 1 Circle

This sangaku requires to determine the relative radii of the circles shown can be solved by an application of the Pythagorean theorem. Find a relationship between the radius of the circle and the side of the square.

6.1 The picture

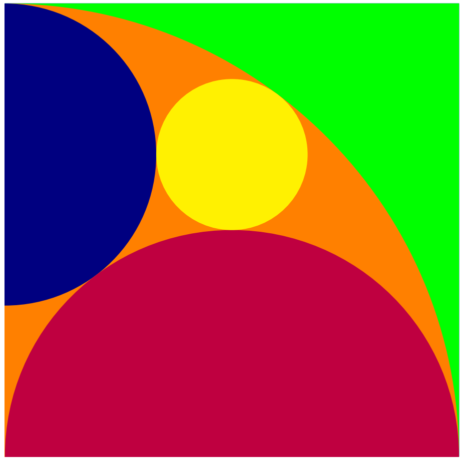


Figure 11: Sangaku 6a in a square

6.2 Explanation and construction

Assume the radius AM equal r and the side of the square is a.

step 1. Firstly, $ME^2 = BE^2 + BM^2$ and the two circles are tangent if $ME = r + \frac{a}{2}$. The equality becomes

$$\left(r+\frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2 + (a-r)^2$$

$$r = \frac{a}{3}$$

we have

$$ME = BK = \frac{5}{6}a$$

step 2.

$$KQ = KE - QE = \frac{2a}{3} - \frac{a}{2} = \frac{a}{6}$$

and

$$MK - \frac{a}{3} = \frac{a}{2} - \frac{a}{3} = \frac{a}{6}$$

The circle K is tangent mutually at the circle E and the circle M.

step 3. The little circle with center K is tangent at the circle B.

$$a - \frac{5a}{6} = \frac{a}{6}$$

step 4. A good method to obtain K is finally to place I such as $DI = \frac{a}{4}$. Therefore, BI intercepts the big circle in G with $GI = \frac{a}{4}$, $FG = \frac{a}{2}$ and FG orthogonal to BG. FG intercepts BC in J such as $DJ = \frac{2a}{3}$. K is the common point between AJ and EF.

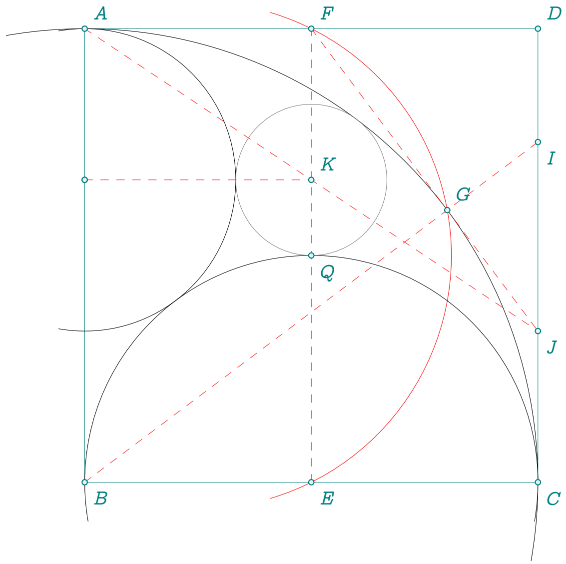


Figure 12: Sangaku 6b

7 Sangaku - Triangle and three circles

References
Weisstein, Eric W. "Circle Inscribing." From MathWorld–A Wolfram Web
http://mathworld.wolfram.com/CircleInscribing.html
Alexander Bogomolny
http://www.cut-the-knot.org/

Construct the figure consisting of a circle centered at O, a second smaller circle centered at O_2 tangent to the first, and an isosceles triangle whose base $[AB]$ completes the diameter of the larger circle $[XB]$ through the smaller $[XA]$. Now inscribe a third circle with center O_3 inside the large circle, outside the small one, and on the side of a leg of the triangle. (from an 1803 sangaku found in Gumma Prefecture)

7.1 The figure

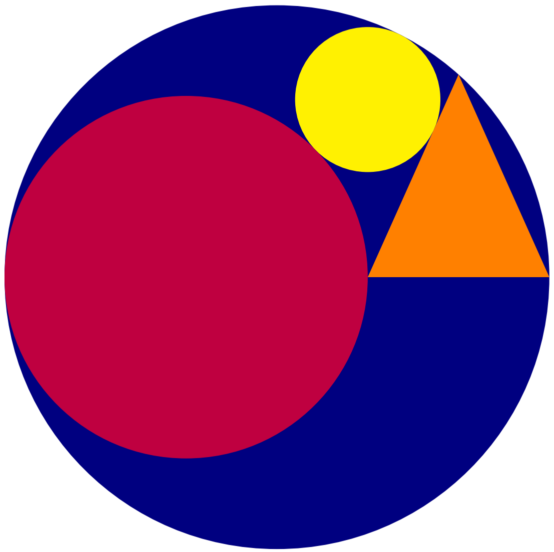


Figure 13: Sangaku - Circle Inscribing

7.2 Explanation

To find the explicit position and size of the circle, let the circle with center O have radius R and be centered at O and let the circle with center O_2 have radius r.

$$(r+a)^2 = r^2 + y^2$$

$$(R-a)^2 = (R-2r)^2 + y^2$$

for a and y gives

$$a = 2r\frac{R-r}{R+r}$$

$$y = AO_3 = 2\sqrt{2Rr}\frac{\sqrt{R-r}}{R+r}$$

but now we need to prove that the circle is tangent to the line AC.
Let α the angle ACD and the angle O_3AC

$$\sin(\alpha) = \frac{AD}{AC}$$

$OD = r$ and $AD = R - r$
OCD is a right triangle with hypotenuse $OC = R$. We have, $OD^2 + CD^2 = OC^2$ by the Pythagorean theorem. In terms of r, the theorem appears as

$$r^2 + CD^2 = R^2$$

which is equivalent to

$$CD^2 = R^2 - r^2$$

and with the right triangle ADC and the Pythagorean theorem

$$AC^2 = AD^2 + CD^2 = (R-r)^2 + R^2 - r^2 = 2R(R-r)$$

finally

$$\sin(\alpha) = \frac{AD}{AC} = \frac{R-r}{\sqrt{2R(R-r)}} = \frac{\sqrt{R-r}}{\sqrt{2R}}$$

Let H the projection point of O_3 on the line AC, and d the length of O_3H

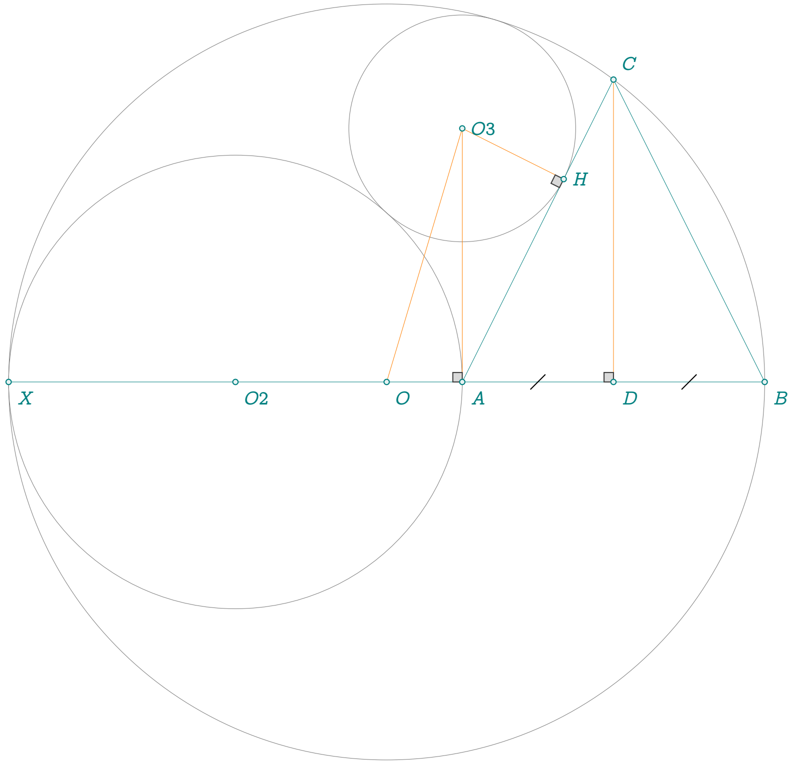
$$\sin(\alpha) = \frac{O_3H}{AO_3} = \frac{d}{y} = \frac{d}{2\sqrt{2R}r\frac{\sqrt{R-r}}{R+r}}$$

Using the two forms of $\sin(\alpha)$

$$\frac{d}{2\sqrt{2R}r\frac{\sqrt{R-r}}{R+r}} = \frac{\sqrt{R-r}}{\sqrt{2R}}$$

So

$$d = 2r\frac{R-r}{R+r} = a$$



8 Two Unrelated Circles

Some references
a Scientific American article by Tony Rothman written in co-operation with Hidetoshi Fukagawa; The book by H. Fukagawa and D. Pedoe.
http://www.sangaku.info/
http://mathworld.wolfram.com
http://www.wasan.jp/english/
http://www.cut-the-knot.org/pythagoras/Sangaku.shtml

8.1 The figure

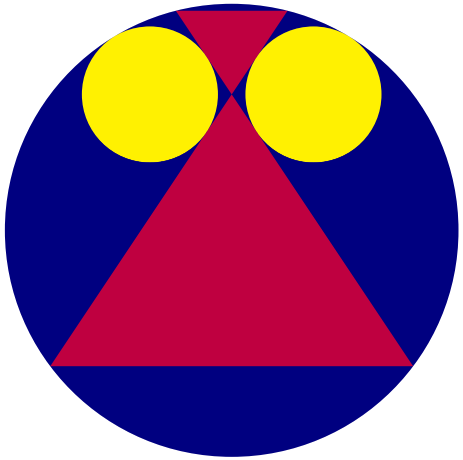


Figure 14: Sangaku 1a Two Unrelated Circles

Chord $[ST]$ is perpendicular to diameter $[CP]$ of a circle with center O at point R. Q is point of $[CP]$ between P and R . $[SQ]$ intersects the circle in V.
Chords $[KN]$ and $[ST]$ are perpendicular to diameter $[CP]$ of a circle with center O at points Q and R. $[SQ]$ intersects the circle in V. (K, S are on one side of $[CP]$, N and T on the other. Q is between P and R.) Let r be the radius of the circle inscribed into the curvilinear triangle TQV. Prove that $\dfrac{1}{r} = \dfrac{1}{PQ} + \dfrac{1}{QR}$.

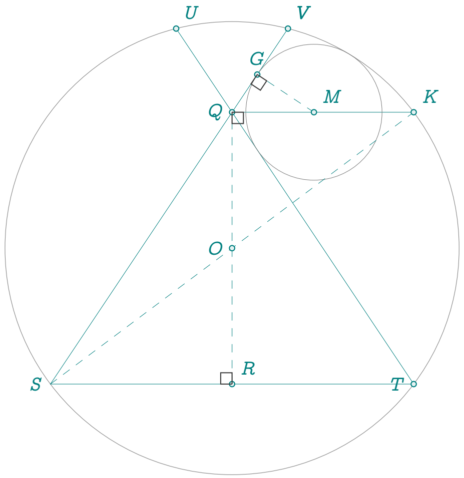


Figure 15: Sangaku 1a+

8.2 Explanation : Idea of Nathan Bowler

Take coordinates such that it is the unit circle $(r = 1)$ with origin M and with Q on the x axis (Q is between O and P). Let $G = (a\ ;\ b)$, $W = (u\ ;\ v)$, $M = (0\ ;\ 0)$ with $a < 0$ and $u > 0$
Then:

- The line (QS) has equation $ax+by = 1$, ($MG = 1$ and (MG) perpendicular to (QS))
- The line (MW) has equation $xv - yu = 0$,
- Q is at $\left(\dfrac{1}{a}\ ;\ 0\right)$,
- O is at $\left(\dfrac{1}{a}\ ;\ \dfrac{v}{au}\right)$,
- $OQ = -\dfrac{v}{au}$,

- $$\begin{aligned} OW^2 &= \left(1-\frac{1}{au}\right)^2 \\ &= \left(u-\frac{1}{a}\right)^2+\left(v-\frac{v}{au}\right)^2 \\ &= u^2-\frac{2u}{a}+\frac{1}{a^2}+v^2-\frac{v^2}{u^2a^2}-\frac{2v^2}{ua} \end{aligned}$$
- $OW^2 = 1-\dfrac{2u}{a}+\dfrac{1}{a^2}-\dfrac{1-u^2}{u^2a^2}-\dfrac{2-2u^2}{ua} = \left(1-\dfrac{1}{ua}\right)^2.$
- S and V are points of circles with center O and M, so if $(x\ ;\ y)$ are coordinates of these points

  $u^2+v^2=1$ and $\left(u-\dfrac{1}{a}\right)^2+\left(v-\dfrac{v}{au}\right)^2=\left(1-\dfrac{1}{ua}\right)^2$

  Further, we obtain

  $$y^2-\frac{av}{u}2y+2\frac{a}{u}-a^2-1=0$$

  The discrimant is $\left(\dfrac{a^2}{u^2}-2\dfrac{a}{u}+1\right)^2=\left(\dfrac{a}{u}-1\right)^2=\left(1-\dfrac{a}{u}\right)^2$

  This equation has two solutions :

  $y_1=\dfrac{av}{u}-\left(1-\dfrac{a}{u}\right)$ negative and $y_2=\dfrac{av}{u}+\left(1-\dfrac{a}{u}\right)$ positive.

Finally

$$\begin{aligned} QR &= -y_1 \\ &= 1-\frac{a}{u}-\frac{av}{u}. \end{aligned}$$

and

$$PQ = OP-OQ = OW-OQ = 1-\frac{1}{ua}+\frac{v}{au}$$
$$PQ = 1-\frac{1}{au}+\frac{v}{au} = 1-\frac{1-v}{au}.$$

So

$$(PQ-1)(QR-1) = \left(-\frac{a}{u}-\frac{av}{u}\right)\left(-\frac{1-v}{au}\right)=\frac{(1-v)(1+v)}{u^2}=1.$$

Equivalently,

$$\frac{1}{PQ}+\frac{1}{QR}=1=\frac{1}{r}.$$

In the general case, some information will be needed. For instance, it is necessary to give the radius of the big circle, the position of points R and Q. I have decided to give $R(0,r)$ relatively to O and Q relatively to P with the value of PQ.

9 Sangaku - Harmonic mean

9.1 Harmonic mean of two numbers

$$\frac{2}{c}=\frac{1}{a}+\frac{1}{b}$$

which says that c is the harmonic mean of a and b.

For two numbers a and b, $\mathcal{A}=\dfrac{a+b}{2}$ and $\mathcal{G}=\sqrt{ab}$, and $\mathcal{H}=\dfrac{2ab}{a+b}=\dfrac{\mathcal{G}^2}{\mathcal{A}}$.

- a and b two numbers such as $OA=a$ and $AB=b$. I is the center of the circle $\mathcal{C}$ with diameter [OB]. [IK] is a radius perpendicular to (OB). It's easy to prove $IK=\mathcal{A}$.
- (AG) is a line perpendicular to (OB) in A and G is point of $\mathcal{C}$. OGB is a right triangle, so

  $$AG^2=OA\times OB$$

  Finally we get $AG=\mathcal{G}=\sqrt{ab}$

GAH and IAG are right rectangles and these rectangles are similars so :

$$\frac{GH}{AG}=\frac{AG}{IH} \text{ and } \frac{GH}{\mathcal{G}}=\frac{\mathcal{G}}{IH}$$

In this case, we have (with $IH=\mathcal{A}$)

$$\mathcal{G}^2=\mathcal{IH}\times GH \text{ and } GH=\frac{\mathcal{G}^2}{\mathcal{A}}$$

Finally

$$GH=\mathcal{H}$$

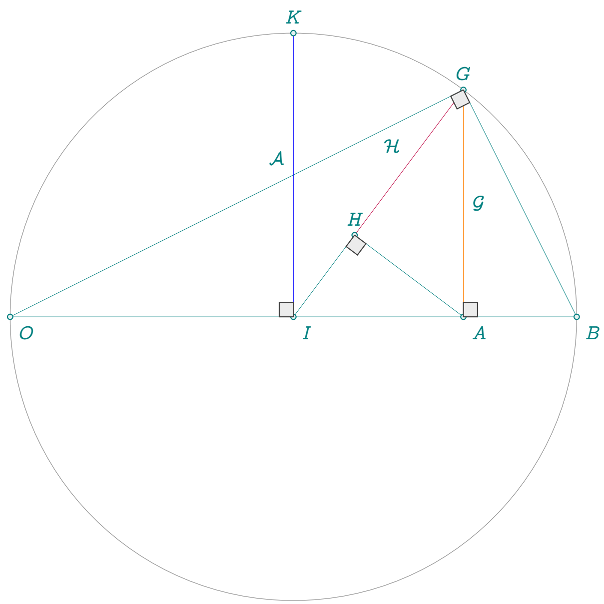


Figure 16: Sangaku 8a

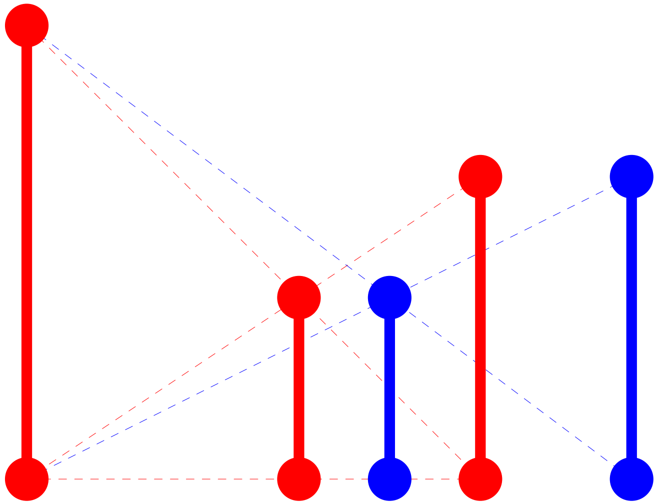


Figure 17: Sangaku 8b

9.2 Sangaku - Harmonic mean

Two vertical segments AI, BJ ($AI = a$ and $BJ = b$), the intersection C of the diagonals is at the height that depends solely on AI and BJ. In fact

$$\frac{1}{CK} = \frac{1}{AI} + \frac{1}{BJ} = \frac{1}{a} + \frac{1}{b}$$

now if $B'J' = BJ$ then $C'K' = CK$.

Let's denote $AI = a$, $BJ = b$, $CK = c$, $IK = \alpha$ and $KJ = \beta$.
The triangles AIJ and CKJ are similar as are triangles BJI and CKI.
We have the proportion

$$\frac{c}{a} = \frac{\beta}{\alpha+\beta} \text{ and } \frac{c}{b} = \frac{\alpha}{\alpha+\beta}$$

We can add the two equalities

$$\frac{c}{a} + \frac{c}{b} = \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta} = 1$$

A division by c gives the desired result:

$$\frac{1}{c} = \frac{1}{a} + \frac{1}{b}$$

which says that c is the half of the harmonic mean of a and b.

10 Sangaku - Three Tangent Circles

10.1 Main method

http://www.cut-the-knot.org/pythagoras/TangentCirclesSangaku.shtml;
[Fukagawa & Pedoe 1.1.2]
Given three circles tangent to each other and to a straight line, express the radius of the middle circle via the radii of the other two. This problem was given as a Japanese temple problem on a tablet from 1824 in the Gunma Prefecture (MathWorld). This problem, too, requires nothing more than a few applications of the Pythagorean theorem. (Alexander Bogomolny)

10.1.1 The figure

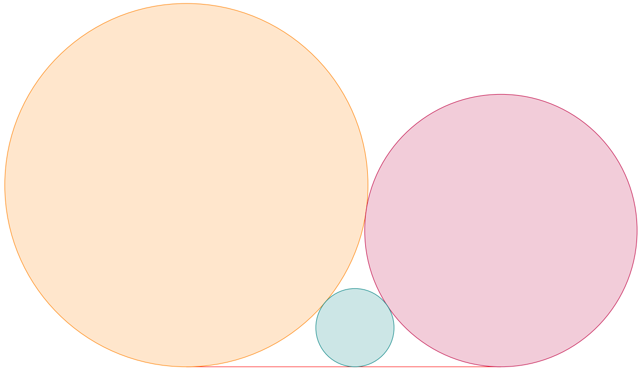


Figure 18: Sangaku 9a - Three Tangent Circles

10.1.2 Lemma

Given two circles tangent to each other and to a straight line at points E and F, we can express EF with the radii (r_a , r_b) of the two circles :

$$EF = 2\sqrt{r_a r_b}$$

Proof : ABH is a right triangle with hypotenuse $AB = r_a + r_b$. We have by the Pythagorean theorem

$$\begin{aligned} AB^2 &= AH^2 + HB^2 \\ &= AH^2 + d^2 \end{aligned}$$

If $r_a > r_b$

$$\begin{aligned} AB^2 &= AH^2 + HB^2 \\ (r_a + r_b)^2 &= (r_a - r_b)^2 + d^2 \end{aligned}$$

Finally $d^2 = 4r_a r_b$ and $d = 2\sqrt{r_a r_b}$.
Now we have two circles ($r_a > r_b$) tangent to each other and to a straight line, and we want to draw another circle tangent to each others and to the same straight line :

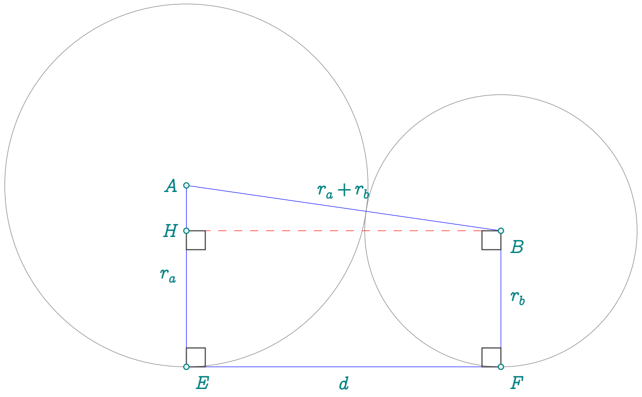


Figure 19: Lemma

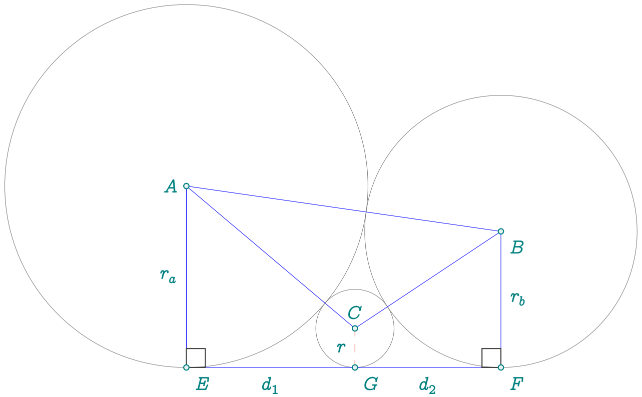


Figure 20: Relation between r, r_a and r_b

We can apply the lemma to the circles $\mathcal{A} = (A, r_a)$ and $\mathcal{B} = (B, r_b)$, then to the circles $\mathcal{A}$ and $\mathcal{C} = (C, r)$ and finally to the circles $\mathcal{B}$ and $\mathcal{C}$.

We get four equations :

$$d = d_1 + d_2$$
$$d = 2\sqrt{r_a r_b}$$
$$d_1 = 2\sqrt{r_a r}$$
$$d_2 = 2\sqrt{r r_b}$$

After simplification, the equations become

$$2\sqrt{r_a r_b} = 2\sqrt{r_a r} + 2\sqrt{r r_b}$$

Divide now by $\sqrt{r}\sqrt{r_a}\sqrt{r_b}$ to obtain

$$\boxed{\frac{1}{\sqrt{r}} = \frac{1}{\sqrt{r_a}} + \frac{1}{\sqrt{r_b}} \quad \text{so} \quad r = \frac{r_a r_b}{(\sqrt{r_a} + \sqrt{r_b})^2}}$$

This is the condition between the radii.

10.1.3 How to make this construction with a ruler and a compass

Before starting the construction let's find out what tools will be needed. We have established a relationship between r, r_a and r_b which is

$$r = \frac{r_a r_b}{(\sqrt{r_a} + \sqrt{r_b})^2}$$

That we can write:

$$r = \frac{r_a r_b}{(r_a + r_b + 2\sqrt{r_a r_b})}$$

So we need to represent $r_a r_b$ and $r_a + r_b + 2\sqrt{r_a r_b}$ before performing a division. This can be achieved in several ways. The only difficulty is to represent $2\sqrt{r_a r_b}$, we will see that it is the length of EF . Here's one:

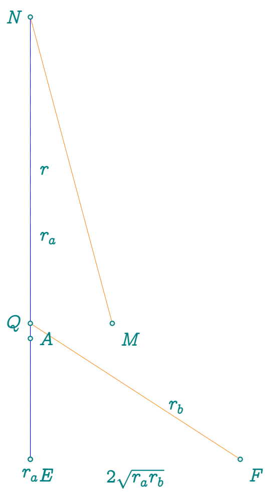


Figure 21: Sangaku 1b How to get r with ruler and compass.

Let's see how to overcome the difficulty of plotting $2\sqrt{r_a r_b}$.

step 1. We draw a line (OX) and a circle with center A tangent to the line in E. The radius of this circle is $EA = r_a = 4$ cm.

step 2. Now we want to draw a second circle with center B of radius $r_b = FB =$ 2 cm mutually tangent to the first circle and to the line (OX) in F.

We place a point P on the line (EA) such as $AP = r_a + r_b$

The circle with diameter AP intercepts the (OX) axis in a point D such as the length $OD = \sqrt{r_a r_b}$.

AED and DEP are right similar triangles and we can write $\dfrac{ED}{EA} = \dfrac{EP}{ED}$, so $ED = \sqrt{r_a r_b}$. It is now easy to obtain F such $EF = 2ED = 2\sqrt{r_a r_b}$. B is a point such as B and P are symmetric with D point of symmetry. We can draw the circle with center B and radius BF.

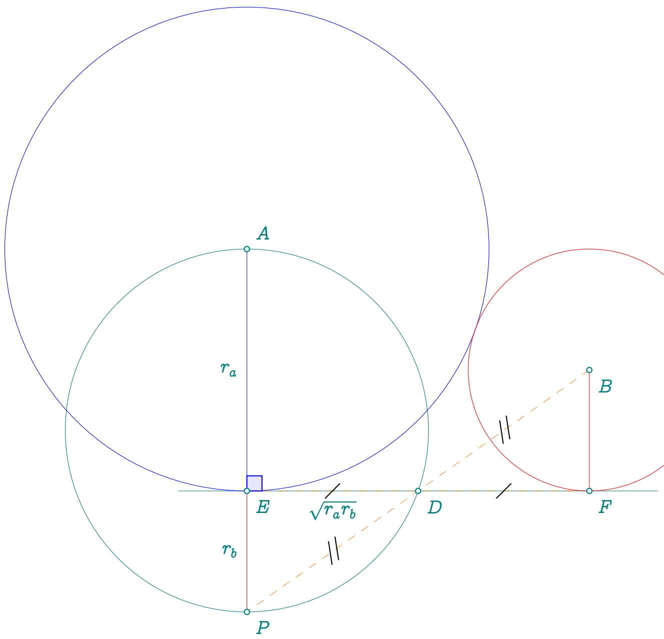


Figure 22: Sangaku 9b $EF = 2\sqrt{r_a r_b}$

step 3. Now we can use the sangaku about harmonic mean.

The line EB intercepts the line AF in a point K such as

$$\frac{1}{KJ}=\frac{1}{EA}+\frac{1}{FB}=\frac{1}{r_a}+\frac{1}{r_b}$$

but $\dfrac{1}{\sqrt{r}}=\dfrac{1}{\sqrt{r_a}}+\dfrac{1}{\sqrt{r_b}}$

so $\dfrac{1}{r}=\dfrac{1}{r_a}+\dfrac{1}{r_b}+\dfrac{2}{\sqrt{r_a r_b}}$

$$\frac{1}{r}=\frac{1}{KJ}+\frac{1}{\dfrac{\sqrt{r_a r_b}}{2}}$$

To find r , we need only to represent $\dfrac{\sqrt{r_a r_b}}{2}$.
$ED=\sqrt{r_a r_b}$, $EF=2\sqrt{r_a r_b}$ and $ED=EF/2$

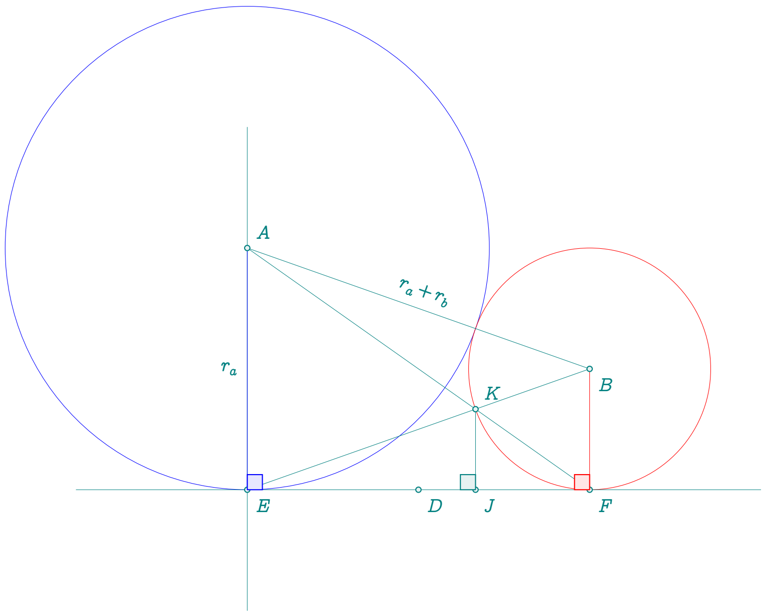


Figure 23: Sangaku 9c

step 4. $AB=r_a+r_b$ and the circles (A,r_a) and (B,r_b) are tangent. The lines (A,F) and (E,B) intersect at a point K. Let J be the orthogonal projection of K on (E,F). The previous sangaku in relation with the harmonic mean allows to write that

$$\frac{1}{KJ}=\frac{1}{EA}+\frac{1}{FB}$$
$$=\frac{1}{r_a}+\frac{1}{r_b}$$

From $\dfrac{1}{\sqrt{r}}=\dfrac{1}{\sqrt{r_a}}+\dfrac{1}{\sqrt{r_b}}$ we get :

$$\frac{1}{r}=\frac{1}{r_a}+\frac{1}{r_b}+\frac{2}{\sqrt{r_a r_b}}=\frac{1}{KJ}+\frac{1}{\dfrac{\sqrt{r_a r_b}}{2}}$$

On the next picture, $EM=EN=\dfrac{ED}{2}=\dfrac{\sqrt{r_a r_b}}{2}$. The line (EK) intercepts the line (JN) at the point R and $IR=r$. Again, we can use the sangaku about harmonic mean.

$$\frac{1}{IR}=\frac{1}{KJ}+\frac{1}{EN}$$

step 5. The point C is the intersection of the circle with center A and radius $a+r$ and the line (SR). In the next picture, $ES=ET$. C is the intersection of the circle (A,AT) and the line (SR). If I is the projection of R on (EF) axis, we have $IR=r$. Let S the projection point of R on (EA) axis.

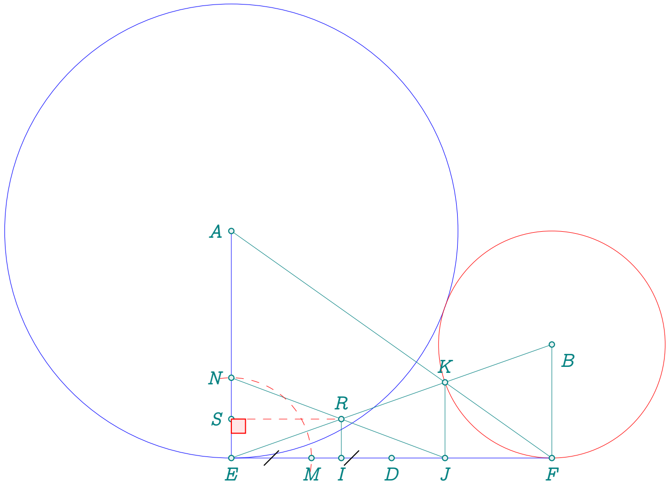


Figure 24: Sangaku 9d $ES = r$

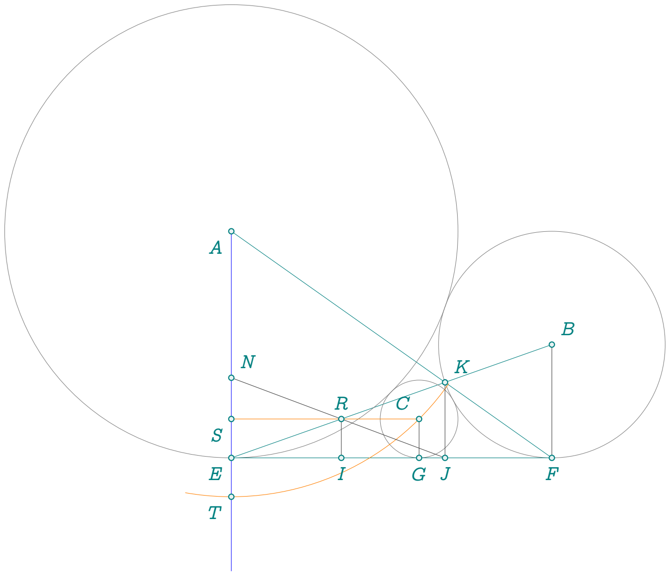


Figure 25: Sangaku 9e First method to get C

10.2 Method Chen Bai

The next contruction is based on a proposition of Chen Bai.
https://math.stackexchange.com/questions/269951/sangaku-how-to-draw-those-three-circles-with-only-a-ruler-and-a-compass/

10.2.1 The figure

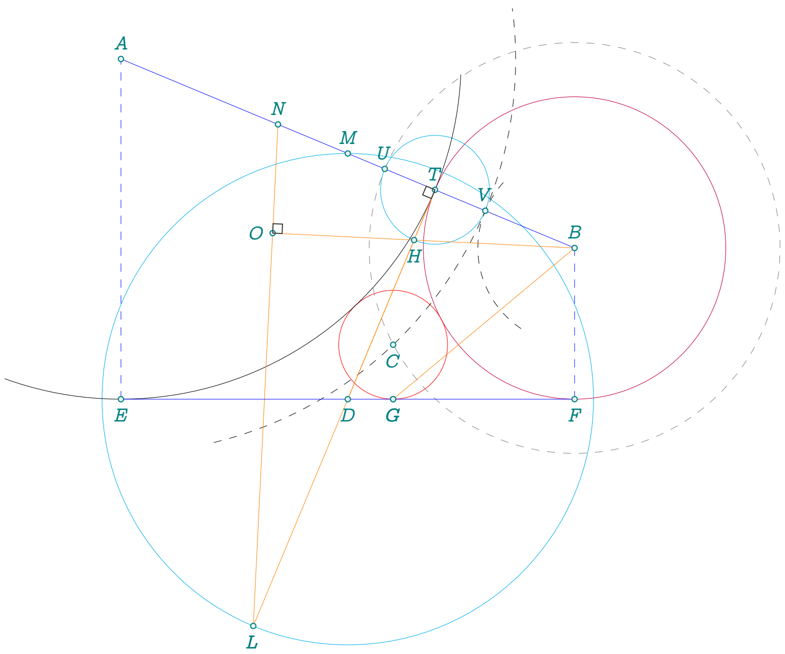


Figure 26: Sangaku 9f: Chen Bai

10.2.2 The explanation

M midpoint of $[AB]$
N midpoint of $[AT]$
L intersection of (TD) with circle (D,M)
O projection of B on (NL)
H intersection of (TD) with (OB)

The triangles BTH et NLT are similar. We have

$$\frac{HT}{BT}=\frac{NT}{LT}$$

$$LT=LD+DT=DM+DT=\sqrt{r_ar_b}+\frac{r_a}{2}+\frac{r_b}{2}=\frac{(\sqrt{r_a}+\sqrt{r_b})^2}{2}$$

$$HT=\frac{BT\times NT}{LT}=\frac{r_b\times r_a/2}{\frac{(\sqrt{r_a}+\sqrt{r_b})^2}{2}}=\frac{r_ar_b}{(\sqrt{r_a}+\sqrt{r_b})^2}$$

$$HT=r=\left(\frac{\sqrt{r_a}\sqrt{r_b}}{\sqrt{r_a}+\sqrt{r_b}}\right)^2$$

10.3 Contruction with inversion

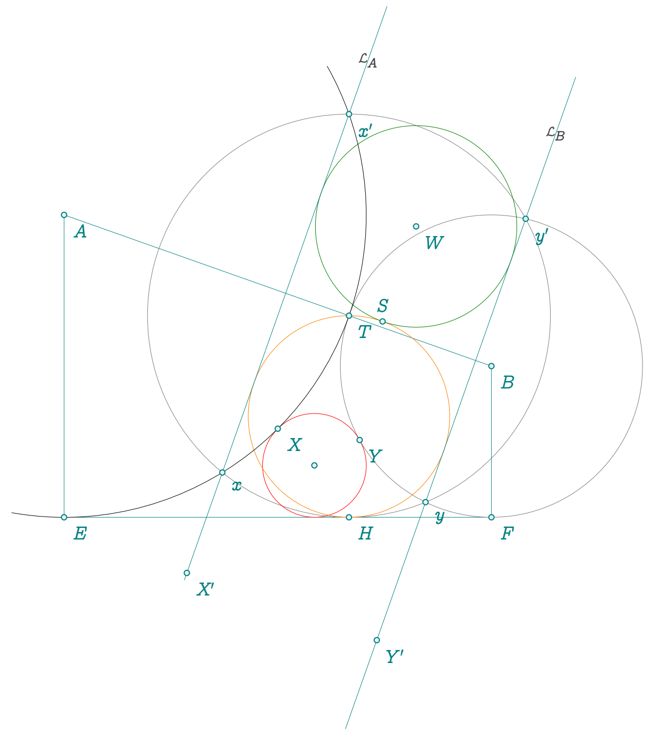


Figure 27: Sangaku 9g: With Inversion

Let us consider the circle $\mathcal{T}$ of center T point common to the circles $\mathcal{A}$ and $\mathcal{B}$ and tangent to the line (EF). It intercepts the two circles at 4 points which define two lines $\mathcal{L}_\mathcal{A}$ and $\mathcal{L}_\mathcal{B}$ perpendicular to the line (AB).
Consider the inversion of center T with respect to the circle $\mathcal{T}$. The circles $\mathcal{A}$ and $\mathcal{B}$ have as inverses the lines $\mathcal{L}_\mathcal{A}$ and $\mathcal{L}_\mathcal{B}$. The orange circle passing through T is tangent to the circle $\mathcal{T}$ and to the line (EF). This circle is therefore tangent to the lines $\mathcal{L}_\mathcal{A}$ and $\mathcal{L}_\mathcal{B}$ since (EF) is tangent to the circles $\mathcal{A}$ and $\mathcal{B}$. Let the green circle $\mathcal{W}$ be tangent to the orange circle $\mathcal{T}$ and to the lines $\mathcal{L}_\mathcal{A}$ and $\mathcal{L}_\mathcal{B}$. It has for inverse a circle tangent to the circles $\mathcal{A}$ and $\mathcal{B}$ and to the line (EF).
The last construction is the most complicated: we look for the intersection points of $\mathcal{W}$ and the circle $\mathcal{T}$. These points are invariant by the inversion and belong to the sought circle. Then we draw the lines from T to X' and Y'. These

straight lines intersect the circles $\mathcal{A}$ and $\mathcal{B}$ at X and Y points of contact with the circle $\mathcal{C}$ sought.

10.4 Other method

The length of the segment $[RM]$ is $r_a + r_b + 2\sqrt{r_a r_b}$. Here is the figure:

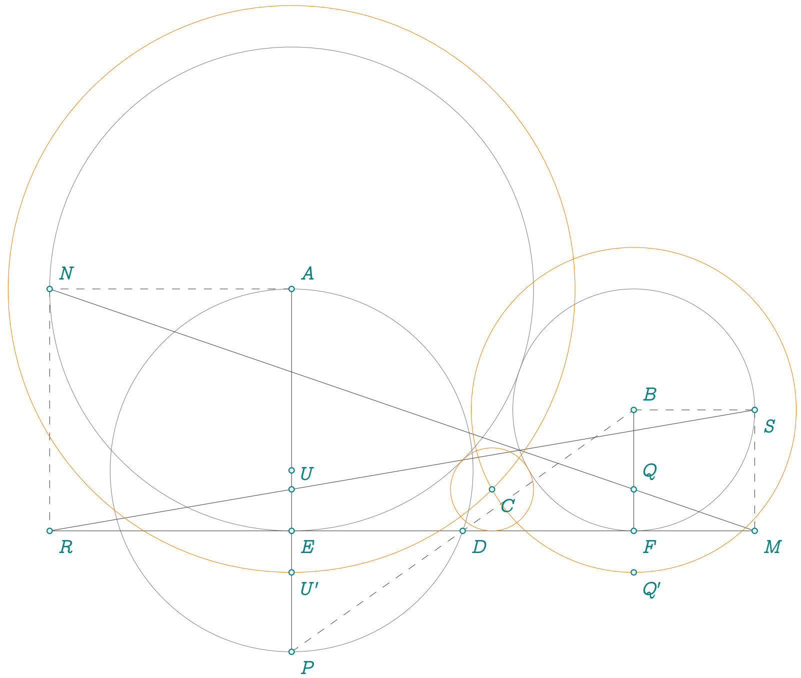


Figure 28: Sangaku 9h

10.5 three-tangent-circles and a square

http://www.gogeometry.com/school-college/3/p1228-three-tangent-circles-line-square.htm

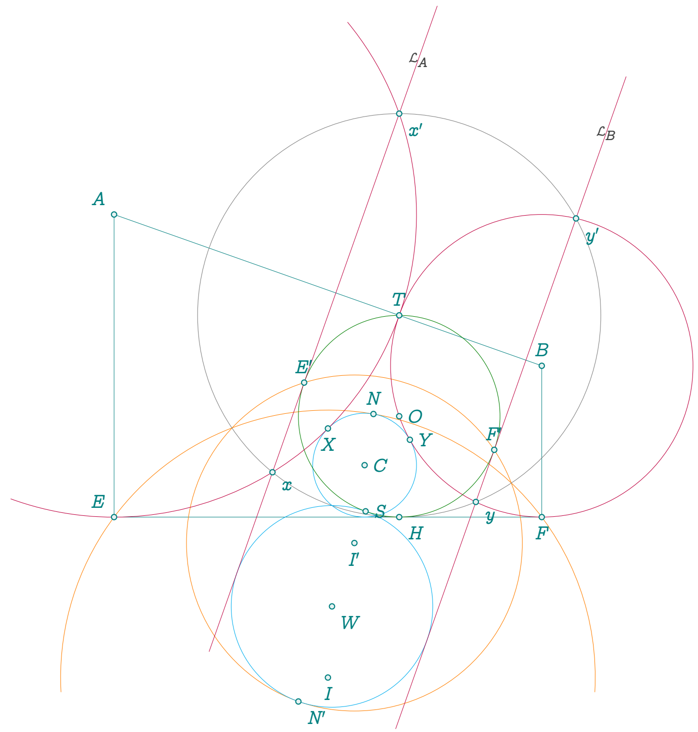


Figure 29: Sangaku 9i: Three circles and square

In the figure 29, circles $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are tangent to each other and to a line L at E, F, K, respectively. The circle $\mathcal{I}$ passes through E and F andtangent to circle $\mathcal{C}$ at T. Tangent to the circle $\mathcal{I}$the midpoint of major arc EF meets AE and BF extended at H and respectively. Prove that $EFGH$ is a square. (by Antonio Gutierrez)

– Step 1. Let's show how to obtain the circle with center I tangent to the circle $\mathcal{C}$ passing through E and F. To do this, consider the circle passing through E', F' and T. E' and F' are the images of E and F by the previous inversion.

They are the points of tangency of the circle passing through T, H and S with the straight lines (xx') and (yy').

The inversion of a circle with diameter $[E'F']$ is a circle passing through E and F and tangent to $\mathcal{C}$ (image of circle with center W by the inversion). W is the point of tangency with the circle of center W passing through S.

– Step 2. Having established the existence of the circle of center I passing through E and F and tangent at N with the circle of center C, let's show that the polygon $EFGH$ is indeed a square.

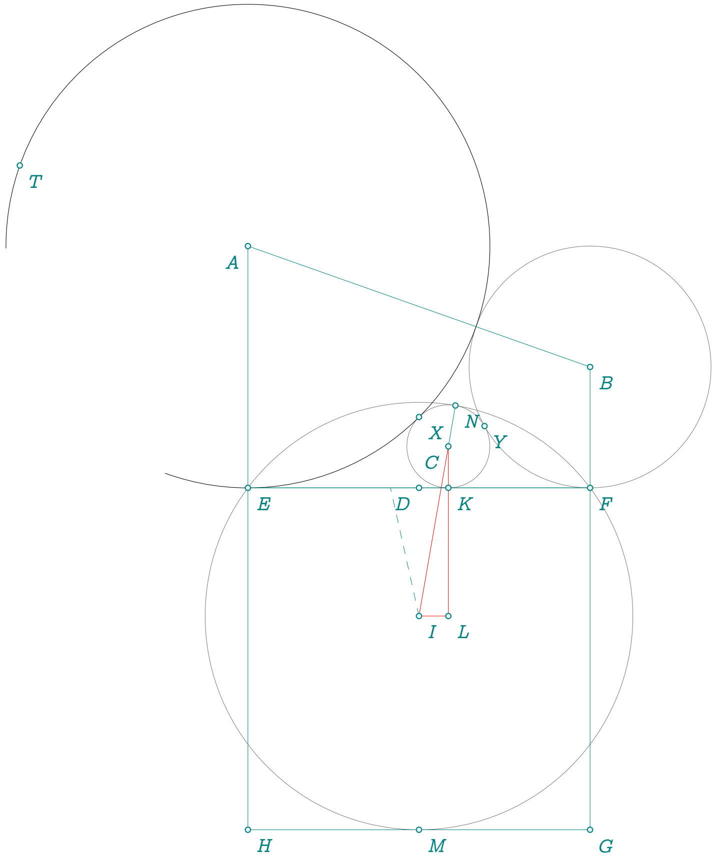


Figure 30: Sangaku 9j

$EFGH$ is a rectangle (obvious)
Demonstration based on Pradyumna Agashe's
(https://gogeometry.blogspot.com/2016/06/geometry-problem-1228-three-tangent.html)
$IC = R - r$
$IL = DK = EK - ED$
and $CL = ID + r$
$IC^2 = CL^2 + IL^2$
$IC^2 = (ID + r)^2 + (EK - ED)^2$
$(R - r)^2 = (LK + r)^2 + IL^2 = (ID + r)^2 + DK^2 = (ID + r)^2 + (EK - ED)^2$
$R^2 - 2rR + r^2 = ID^2 + 2rID + r^2 + (EK - ED)^2$
$R^2 - 2rR = ID^2 + 2rID + (EK - ED)^2$

or $R^2 = IE^2 = ED^2 + ID^2$

we get $ED^2 + ID^2 - 2rR = ID^2 + 2rID + (EK - ED)^2$
$ED^2 - 2rR = 2rID + (EK - ED)^2$
$2rID + 2rR = ED^2 - (EK - ED)^2$
$2r(ID + R) = ED^2 - (EK - ED)^2$
$2r(ID + R) = 2EK \cdot ED - EK^2$
$2r(ID + R) = EK \cdot (2ED - EK) = EK \cdot KF$
$2r(ID + R) = 2\sqrt{ar} \cdot 2\sqrt{br} = 4r\sqrt{ab} = 2rEF$
$HE = ID + R = EF$
Then EFGH is a square

10.6 Three-tangent-circles and 3-4-5-triangle

http://www.gogeometry.com/school-college/3/p1229-three-tangent-circles-line-3-4-5-triangle.htm

In the figure below, circles A, B, and C are tangent to each other and to a line (EF) at E, F, and K, respectively. The circle I passes through E and F and is tangent to circle C. BF extended meets circle I at D. Prove that triangle DEF is a 3-4-5 right triangle.

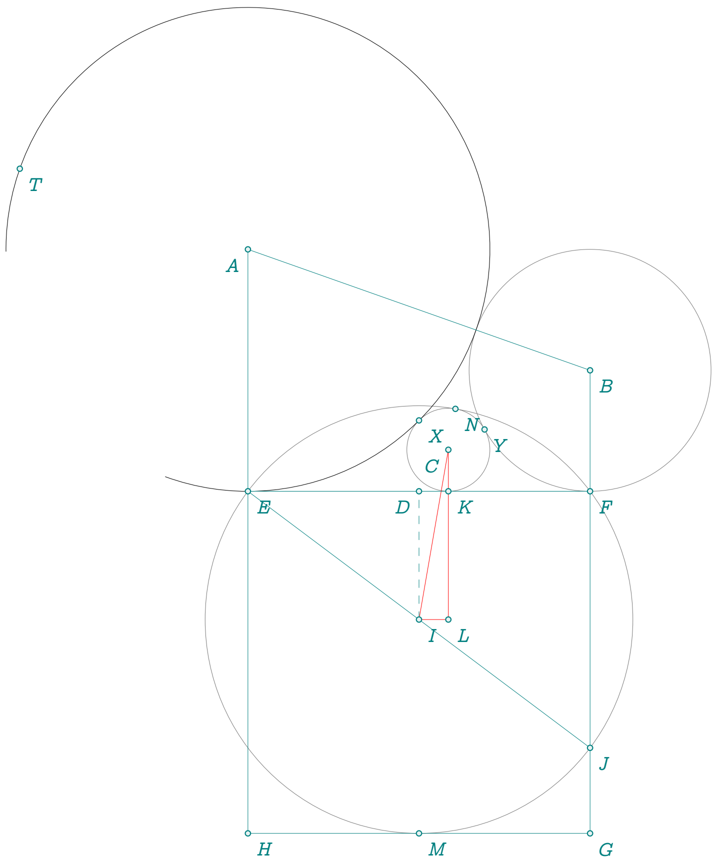


Figure 31: 3-4-5-triangle

(Solution : Sumith Peiris July 5, 2016)
Let $FJ = x$, $EJ = y$ and $EF = z$ and further the circles have radii r_a,r_b,r and $R = y/2$
For right triangle EFG $x^2 + z^2 = y^2$
For right triangle with hypotenuse ILC
$IC^2 = IL^2 + LC^2 = DK^2 + LC^2$
$(y/2 - r)^2 = (x + 2r)^2 + (2EK - z)^2$
$(y - 2r)^2 = (x/2 + r)^2 + (EK - z/2)^2$
$(x + y)r = EK(z - EK) = 4r\sqrt{r_a r_b}$

But $EK = 2\sqrt{r_a r}$, $KF = z - EK = 2\sqrt{r_b r}$ and $z = 2\sqrt{r_a r_b}$
Hence $x + y = 4\sqrt{r_a r_b} = 2z$
Hence $x + y = 2z$ and $x^2 + z^2 = y^2$
We can eliminate z to find y as a function of x.
$y^2 - x^2 = (x + y)^2/4$ which yields $y - x = (x + y)/4$
So $3y = 5x$ which involves $3z = 4x$.
Posons $x = 3k$ where k is a real number then
$x = 3k$ $y = 5k$ and $z = 4k$. The lengths of the sides are proportional to 3, 4 and 5.
Therefore JEF is a 3-4-5 triangle

10.7 A Sangaku with an Egyptian Attachment

Alexander Bogomolny
www.cut-the-knot.org/Curriculum/Geometry/VariableTangentSangaku.shtml
Assume points E and F are fixed on a line EF and two circles are drawn touching EF at E and F and tangent to each other. A circle C is tangent to EF and the two circles externally. Prove that, as the two circles change, circle C remains tangent to a fixed circle through E and F. Moreover, the radius of the latter is $\frac{5}{8}EF$.

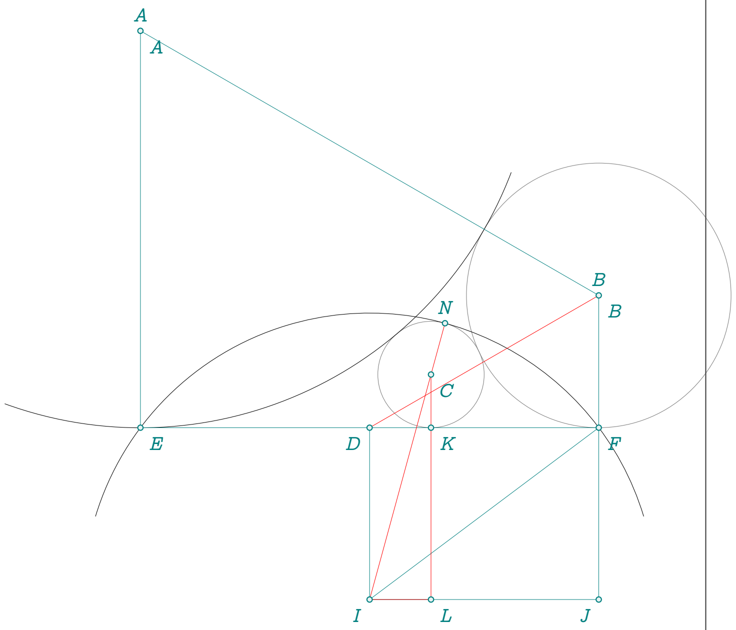


Figure 32: Egyptian triangle

In the previous section we saw that the triangle EFG is an Egyptian triangle. Let J be the intersection of the line IL with the line BF. DIF is similar to EFG and is therefore also an Egyptian triangle, as is DIF. Thus there exists a real t such that $DF = 4t$ and $IF = 5t$. Then $DF = EF/2$, so $t = EF/8$. we easily deduce that $IF = \frac{5}{8}EF$. So the circle of center I passing through F is fixed.

10.8 Special position

If $r_a = 4$ and $r_b = 1$ then X' is on the line (EF), X is the intersection of $\mathcal{A}$, $\mathcal{C}$, (TX') and $\mathcal{O}(O, H)$. X is then the middle of $[TX']$.

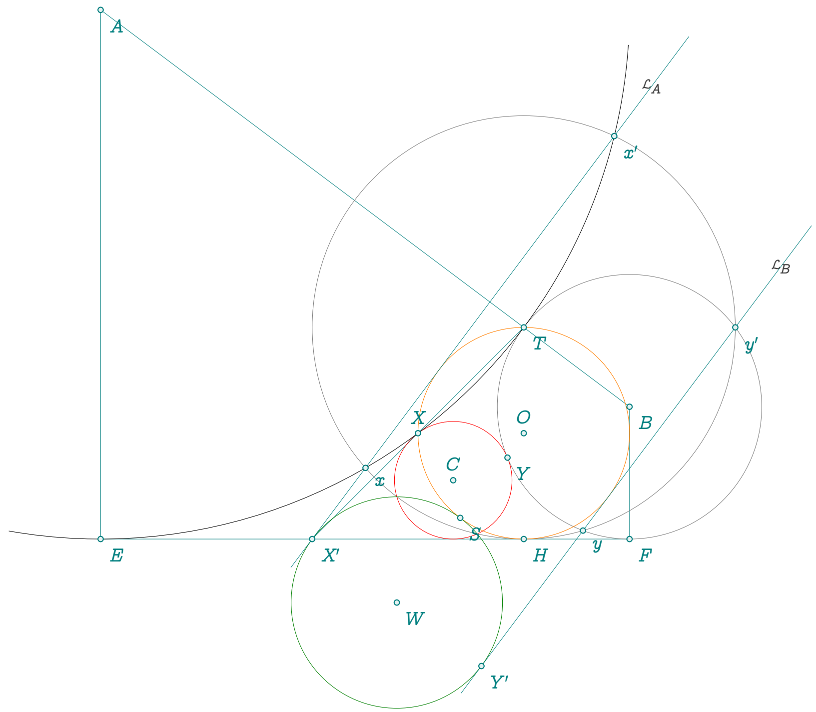


Figure 33: Sangaku 9m: Special position