THE DROZ–FARNY LINE THEOREM

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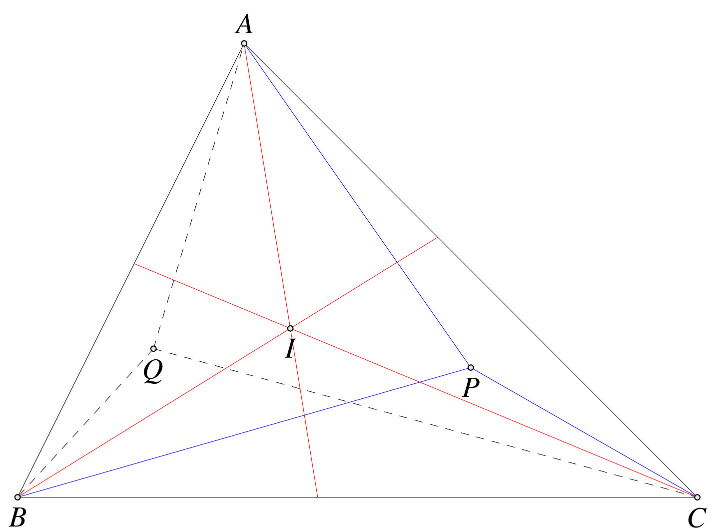
ABSTRACT. This article is devoted to a classical and elegant result in Euclidean geometry: the Droz–Farny line theorem. The presentation is inspired by the paper *Back to Euclidean Geometry: Droz–Farny Demystified* by Titu Andreescu and Cosmin Pohoata, as well as by the Wikipedia page devoted to the theorem. For simplicity, we assume that the point lies inside the triangle.

0.1. Isogonal conjugate in a triangle.



Definition 1 (Isogonal conjugate)**.** *In a triangle ABC, the* isogonal conjugate *of a point P is constructed by reflecting the lines PA, PB, and PC with respect to the internal angle bisectors of $\widehat{A}$, $\widehat{B}$, and $\widehat{C}$. The reflected lines concur at a point called the isogonal conjugate of P. This construction applies to points not lying on the sidelines of the triangle and follows from the trigonometric form of Ceva's theorem.*

Note: The isogonal conjugate of the orthocentre H is the circumcentre O.

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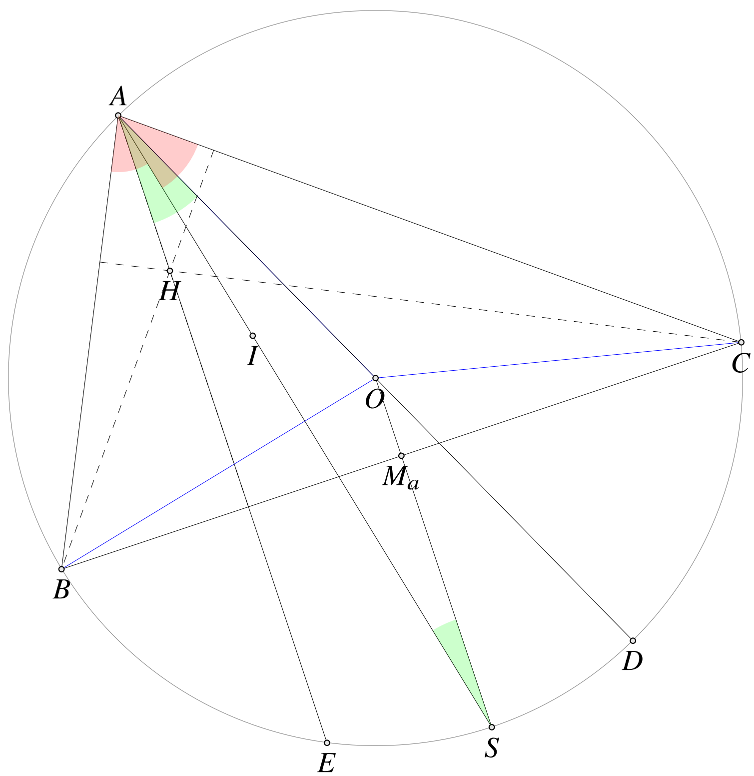
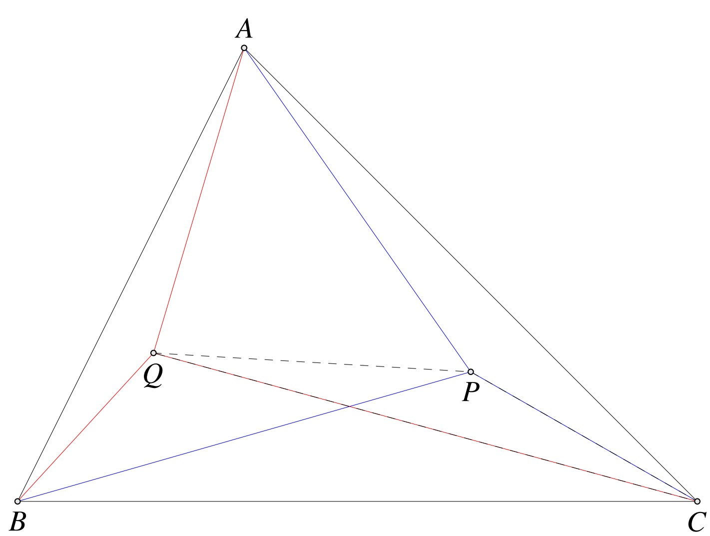


Proof. First, O, S, and M_a are collinear. Indeed, the internal bisector of $\widehat{BAC}$ meets the circumcircle again at S, the midpoint of arc $\widehat{BC}$ not containing A. This midpoint lies on the perpendicular bisector of segment BC, just like M_a and O; hence the three points are collinear.

Moreover, $\widehat{EAS} = \widehat{ASO}$, so line (AO) is the reflection of line (AH) across the bisector of $\widehat{BAC}$. Applying the same argument at the other two vertices shows that O is the isogonal conjugate of H. □

1. Some lemmas

1.1. Angular relationship related to isogonal conjugate points.



Lemma 1. *Let P be a point in a non-degenerate triangle ABC, and let Q be the isogonal conjugate of P with respect to ABC. Then*

$$\widehat{BQC} + \widehat{BPC} = 180^\circ + \widehat{BAC}.$$

Proof. Since Q is the isogonal conjugate of P, we have

$$\widehat{QBC} = \widehat{PBA}, \qquad \widehat{QCB} = \widehat{ACP}.$$

Hence in triangle BQC,

$$\widehat{BQC} = 180^\circ - \widehat{PBA} - \widehat{ACP}.$$

In triangle BPC,

$$\widehat{PBC} = \widehat{ABC} - \widehat{PBA}, \qquad \widehat{PCB} = \widehat{ACB} - \widehat{ACP},$$

so

$$\widehat{BPC} = 180^\circ - \widehat{ABC} - \widehat{ACB} + \widehat{PBA} + \widehat{ACP}.$$

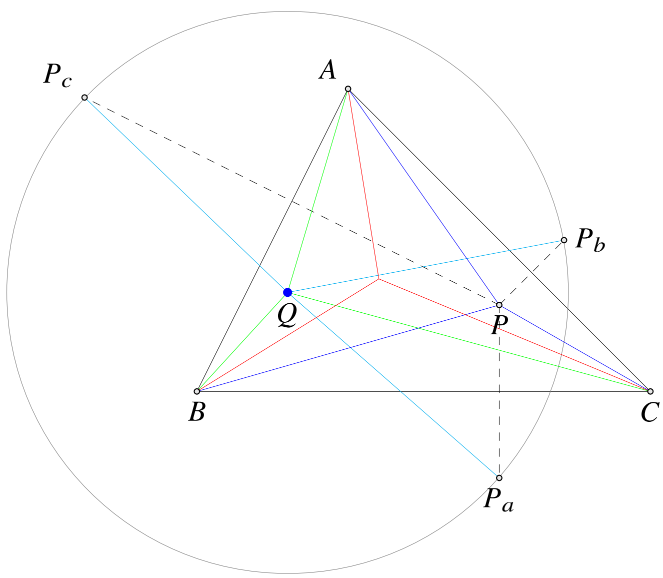
Adding the two expressions gives

$$\widehat{BQC} + \widehat{BPC} = 180^\circ + \widehat{BAC}.$$

□

1.2. Another definition or construction of a isogonal conjugate.

Lemma 2. *Let P be a point in a non-degenerate triangle ABC. Let P_a, P_b, and P_c be the reflections of P across the sidelines (BC), (CA), and (AB), respectively. Then the circumcenter of triangle $P_aP_bP_c$ is the isogonal conjugate of P with respect to triangle ABC.*



Proof. Let Q be the isogonal conjugate of P. By definition, the pairs of lines $(PA),(QA)$, $(PB),(QB)$, and $(PC),(QC)$ are symmetric with respect to the angle bisectors of A, B, and C.

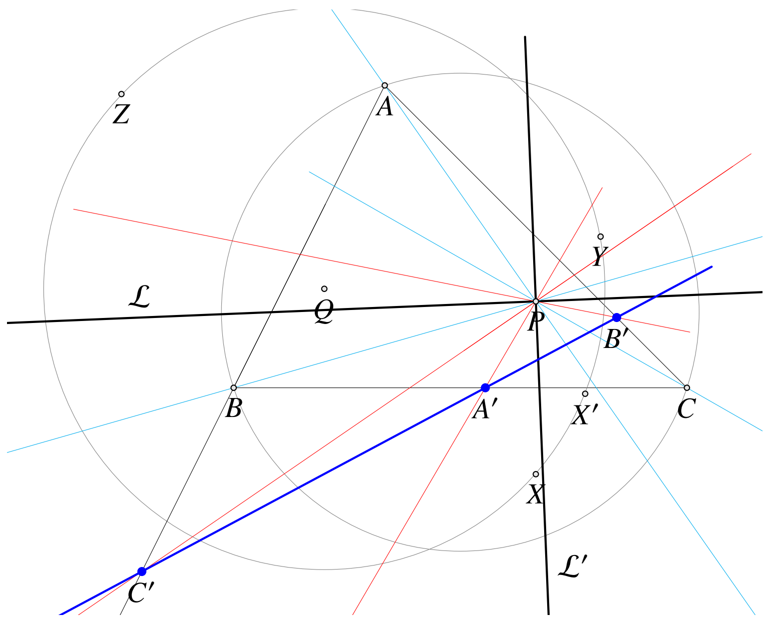
Reflecting P across (BC) sends (PA) and (PB) to lines symmetric with respect to (BC). Hence P_a lies on the reflections of both (PA) and (PB), which implies that $QP_a = QA = QB$. Similarly one proves that $QP_b = QB = QC$ and $QP_c = QC = QA$.

Thus Q is equidistant from P_a, P_b, and P_c, hence it is the circumcenter of triangle $P_aP_bP_c$. □

Note: Since Q is the circumcenter, it is enough to prove that $CP_a = CP = CP_b$. Hence C lies on the perpendicular bisector of chord P_aP_b, and therefore $(QC) \perp (P_aP_b)$.

1.3. Goormaghtigh's generalization

Theorem 1 (Goormaghtigh's generalization)**.** *Let P be a point inside a triangle ABC. Let a line $\mathcal{L}$ through P be given. Reflect the lines (PA), (PB), and (PC) across $\mathcal{L}$; let their intersections with the sides (BC), (CA), and (AB) be denoted by A', B', and C', respectively. Then the points A', B', and C' are collinear.*



Proof. The proof given below is inspired by the article: *Back to Eucldiean Geometry: Droz-Farny Demystified*.

Let Q be the isogonal conjugate of P with respect to triangle ABC, and let Q' be the isogonal conjugate of P with respect to triangle $AB'C'$. Since P is not on the circumcircle of ABC, the point Q is finite.

Claim. $Q = Q'$.

Proof of the claim. Since P and Q are isogonal conjugates in triangle ABC, we have with lemma (**??**)

$$\widehat{BQC} + \widehat{BPC} = 180^\circ + \widehat{BAC}.$$

Similarly, since P and Q' are isogonal conjugates in triangle $AB'C'$, we get

$$\widehat{C'Q'B'} + \widehat{B'PC'} = 180^\circ + \widehat{C'AB'}$$

But $\widehat{BAC} = \widehat{C'AB'}$ and $\widehat{CPB} = \widehat{B'PC'}$ (because (PB') and (PC') are the refection of (PB) and (PC) across $\mathcal{L}'$), hence

$$\widehat{BQC} = \widehat{C'Q'B'}. \tag{1}$$

Let X, Y, Z be the reflections of P across (BC), (CA), (AB), respectively, and let X' be the reflection of P across $B'C'$.

Then $\widehat{ZXY} = \widehat{BQC}$,(because (QC) is orthogonal to (XY) and (QB) is orthogonal to (XZ)).

wheras $\widehat{ZX'Y} = \widehat{C'Q'B'}$ (because $(Q'B')$ is orthogonal to $(X'Y)$ and $(Q'C')$ is orthogonal $(X'Z)$).

By (1) we obtain $\widehat{ZXY} = \widehat{ZX'Y}$, hence X, Y, Z, X' are concyclic.

By the lemma (**??**), the circumcenter of triangle XYZ is Q, while the circumcenter of triangle $X'YZ$ is Q'. Since these two triangles lie on the same circle, their circumcenters coincide, and therefore $Q = Q'$.

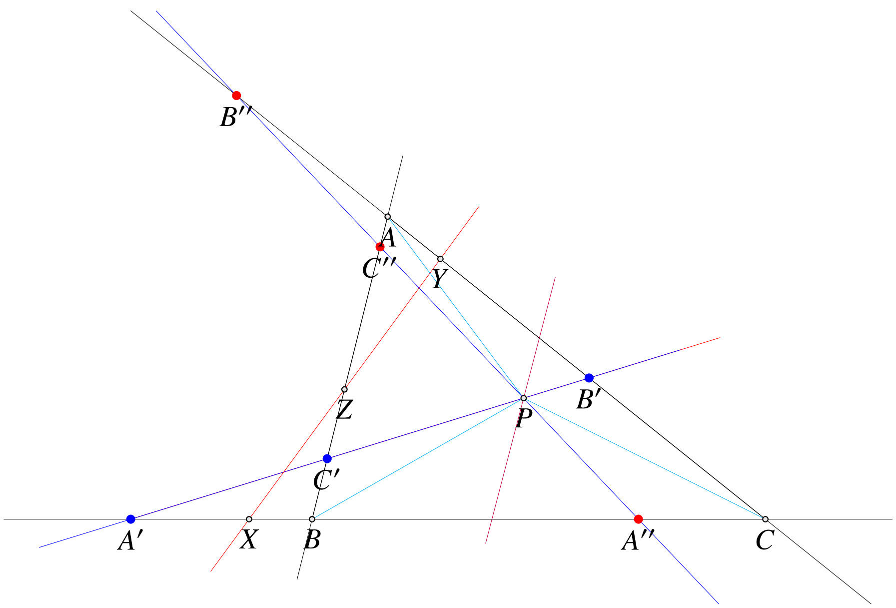
Since Q is also the isogonal conjugate of P with respect to triangles $A'BC'$, $AB'C$, and $A'B'C$, the same argument applied cyclically shows that

$$\widehat{BC'A'} = \widehat{AC'B'},$$

which proves that A', B', C' are collinear. This completes the proof. ◻

1.4. Droz-Farny's generalization

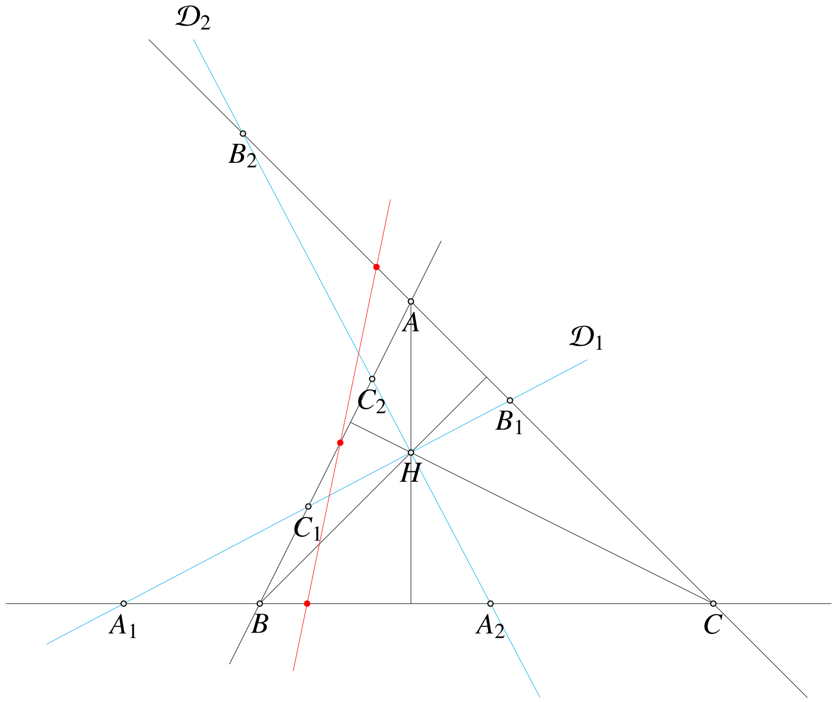
Theorem 2 (Droz-Farny's generalization)**.** *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be lines passing through a given point P in the plane of a triangle ABC. Let A', B', C' and A'', B', C' be the intersections of $\mathcal{L}_1$ and $\mathcal{L}_2$ with (BC), (CA), (AB), respectively. Furthermore, let X be the intersection of (BC) with the reflection of (AP) into the internal angle bisector of $\widehat{A'PA''}$, and similarly, define Y and Z. Then points X, Y, Z are collinear.*



Proof. Apply Theorem 1 to the internal (or external) bisector of $\widehat{A'PA''}$. The result follows immediately. ◻

1.5. Droz-Farny

Theorem 3 (Droz–Farny)**.** *Let ABC be a triangle with orthocenter H. Two perpendicular lines are drawn through H. These lines intersect the sidelines (BC), (CA), and (AB) at six points, forming three segments on the sidelines. The midpoints of these three segments are collinear.*



Proof. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two perpendicular lines through the orthocenter H of triangle ABC. Denote by

$$A_1, B_1, C_1 \quad \text{and} \quad A_2, B_2, C_2$$

the intersections $\mathcal{D}_1$ and $\mathcal{D}_2$ with (BC), (CA), (AB), respectively.

Since H is the orthocenter of ABC, the lines (AH), (BH), and (CH) are the altitudes of ABC. In particular,

$$(AH) \perp (BC), \qquad (BH) \perp (CA), \qquad (CH) \perp (AB).$$

Now observe that (BC) is also the line (A_1A_2), hence $(AH) \perp (A_1A_2)$. Thus (AH) is an altitude of triangle (A_1HA_2). Similarly, (BH) and (CH) are altitudes of triangles (B_1HB_2) and (C_1HC_2), respectively.

Let O_a, O_b, and O_c denote the circumcenters of triangles A_1HA_2, B_1HB_2, and C_1HC_2. Since in any triangle the circumcenter is the isogonal conjugate of the orthocenter, the reflections of the altitudes (AH), (BH), and (CH) across the internal angle bisectors of triangles A_1HA_2, B_1HB_2, and C_1HC_2 pass through O_a, O_b, and O_c, respectively.

But reflecting (AH) across the bisector of $\widehat{A_1HA_2}$ gives the line symmetric to (AH) with respect to this angle; since $(A_1H) \perp (A_2H)$, this reflection is parallel to (A_1A_2). Hence O_a lies on the perpendicular bisector of $[A_1A_2]$, and therefore O_a is the midpoint of segment $[A_1A_2]$.

The same argument shows that O_b and O_c are the midpoints of $[B_1B_2]$ and $[C_1C_2]$, respectively. These are precisely the Droz–Farny midpoints. $\square$