

**THIS DOCUMENT BRINGS TOGETHER A VARIETY OF GEOMETRIC
RESULTS, INCLUDING DEFINITIONS, THEOREMS, LEMMAS,
APPLICATIONS, AND NUMEROUS EXAMPLES.**

ALAIN MATTHES

ABSTRACT. All figures are produced using [tkz-elements](#) and [tkz-euclide](#).

You will find this document, along with many additional examples, on my website: [al-termundus.fr](#).

Please report any typos or comments concerning this documentation to [Alain Matthes](#).

CONTENTS

Part 1. Foundations	5	9.2. Excircles	22
1. Notation and Basic Objects	5	9.3. Example: Cevian circle with orthocenter	22
1.1. Notation	5	10. Orthogonal Circles	22
2. Collinearity and Ratios	5	10.1. Angle of two intersecting circles	22
2.1. Fundamental Theorems	5	10.2. Orthogonal circles	24
2.2. Menelaus' theorem (weak version)	6	10.3. Circle orthogonal to two given circles	25
3. Parallelism and Similarity	6	10.4. Orthogonal Circles – Secant Criterion	25
3.1. Definitions	6	11. Power of a Point and Radical Axis	26
3.2. Definition in Affine Geometry	7	11.1. Power of a point with respect to a circle	27
3.3. Basic Results	7	11.2. Properties of the power	27
4. Basic Constructions	7	11.3. Powers of a point with respect to two circles	28
4.1. Definitions	7	11.4. Pencil of circles	28
4.2. Fundamental Properties	7	11.5. Fixed point pencil	28
4.3. Length transfer	7	11.6. Radical axis	29
4.4. Gold division	8	11.7. Construction of the radical axis for two non-intersecting circles	31
4.5. Gold ratio with segment	9	11.8. Radical axis of circumcircle and Apollonius circle	31
Part 2. Triangle Geometry	9	11.9. Circle in circular segment	31
5. Basic Elements	9	11.10. Circle and path	32
5.1. Perpendicular Bisectors or Mediators	9	11.11. Symmedian and tangents	32
5.2. Altitudes	9	11.12. Circle through incircle	33
5.3. Medians	10	11.13. Euler's relation	33
5.4. Bisectors	11	Part 4. Inversion	34
5.5. Angle Bisector Theorem: Internal angle bisector	12	12. Inversion	34
5.6. Angle Bisector Theorem: External angle bisector	12	12.1. Definition	34
5.7. Specific lines of a triangle	13	12.2. Inversion d'un cercle orthogonal	35
6. Triangle Centers	14	12.3. Cocyclic points	35
6.1. Existence of the Euler line	14	13. Midcircle	35
7. Special Configurations	14	13.1. Construction of the midcircle of two disjoint circles	38
7.1. Lemoine	14	13.2. Midcircles and Arbelos	39
7.2. Orthopole and Simson line	15	Part 5. Special Circles	41
7.3. Lemoine point	15	14. Apollonius Circles	41
Part 3. Circle Geometry	16	14.1. Apollonius circles of a triangle	41
8. Angles and Circles	16	14.2. Apollonius circles in a triangle with method	42
8.1. Apollonius circle	16	14.3. Apollonius circles and circumcircle of a triangle	43
8.2. South Pole	16	15. Soddy Circles	43
8.3. Inscribed Angles	17	15.1. Soddy circles of a triangle	43
8.4. Tangent and circle	18	15.2. Soddy circle without function	43
8.5. Tangent-Chord Theorem	18	15.3. Soddy circle with function	44
8.6. Alternate Segment Theorem	21	15.4. Construction of Soddy's circles	44
9. Classical Circles	21	16. Various circles	46
9.1. Circle-Point Midpoint Theorem	21	16.1. First Lemoine circle	46
		16.2. First and second Lemoine circles	46
		16.3. Adams's circle	47

16.4.	Van Lamoen's circle	47	23.5.	Ellipse with Foci and a Point	66
16.5.	Yiu's circles variant one	47	24.	Parabola	67
16.6.	Yiu's circles variant two	48	24.1.	Parabola with Focus, Axis of Symmetry, and a Point on the Curve	67
16.7.	Yff's circles variant one	48	25.	Conics in a Triangle	67
16.8.	Yff's circles variant two	48	25.1.	Orthic_inellipse with search_ellipse	67
17.	Arbelos and Related Figures	49	25.2.	Kiepert hyperbola	68
17.1.	Gold Arbelos	49	25.3.	Kiepert parabola	68
17.2.	Bankoff circle	49	25.4.	Euler ellipse	69
17.3.	Related circles from Leon Bankoff	51			
17.4.	Gold Arbelos properties	51	Part 9. Advanced Theorems		71
17.5.	Pentagons in a golden arbelos	52	26.	Feuerbach Theorem	71
18.	Advanced Circle Configurations	52	26.1.	Feuerbach theorem	71
18.1.	Six circles in a triangle	52	26.2.	Feuerbach_Axel	71
18.2.	Pappus chain	54	26.3.	Feuerbach_Hioux	74
18.3.	Three Circles	54	27.	Morley Triangle	74
Part 6. Projective Geometry		55	27.1.	Morley triangle	74
19.	Harmonic Division	55	28.	Thébault Problems	74
19.1.	Definitions	55	28.1.	Thébault's problem III	74
19.2.	Construction of a harmonic division	56	28.2.	Thébault's problem II	75
19.3.	Harmonic conjugate of a point outside a segment	56	28.3.	Thébault's problem III with a method	75
19.4.	Harmonic division with golden ratio	56	29.	The Apollonius Circle of a Triangle	77
19.5.	Bisector and harmonic division	57	29.1.	Apollonius circle v1 with inversion	77
20.	Pascal's Theorem	57	30.	Classical Theorems	79
20.1.	Definitions	57	30.1.	Reuschle's theorem	79
20.2.	Main Result	57	30.2.	Viviani's Theorem	79
20.3.	Dual Statement	58	30.3.	Iran Lemma	80
20.4.	Degenerate Case	60	30.4.	Reim's theorem 1	80
20.5.	Application: Pascal' theorem weak version	61	30.5.	Reim's theorem 2	81
20.6.	Pascal's theorem with a triangle	62	30.6.	Three chords	82
20.7.	Application of Pascal's theorem	62	30.7.	Altshiller-Court's theorem	83
21.	Poles and Polars	64	30.8.	D'Alembert's Theorem	83
21.1.	Polar and Pascal's theorem	64	30.9.	Archimedes' Book of Lemmas: Proposition 1	84
Part 7. Polygons		64	31.	Apollonius' problem	85
22.	Parallelograms	64	Overview		85
22.1.	Varignon's Theorem	64	Unified API (paths everywhere)		85
22.2.	Wittenbauer's Parallelogram	65	Using <code>tkz.nodes_from_paths</code> instead of <code>\tkzDrawCirclesFromPaths</code>		85
Part 8. Conics		65	31.1.	Class point: method PPP(p, p)	86
23.	Ellipse	65	31.2.	Class line: method LPP(p, p)	87
23.1.	Ellipse	65	31.3.	Class line: method LLP(L, p)	88
23.2.	Ellipse with radii	65	31.4.	Class line: method LLL(L, L)	90
23.3.	Ellipse_with_foci	65	31.5.	Class: method CLL(L, L)	91
23.4.	Ellipse with Center, Vertex, and Covertex	66	31.6.	Class circle: method CPP(p, p)	93
			31.7.	Class circle: method CLP(L, p, mode)	94
			31.8.	Class circle: method CCP(C, p, mode)	96
			31.9.	Class circle: method CCL(C2, L)	98

31.10. Case circle: method CCC(C2, C3 [, opts])	99
31.11. Class circle: method CCC_gergonne(C_2, C_3)	101
References	103

Part 1. Foundations

1. Notation and Basic Objects

1.1. Notation

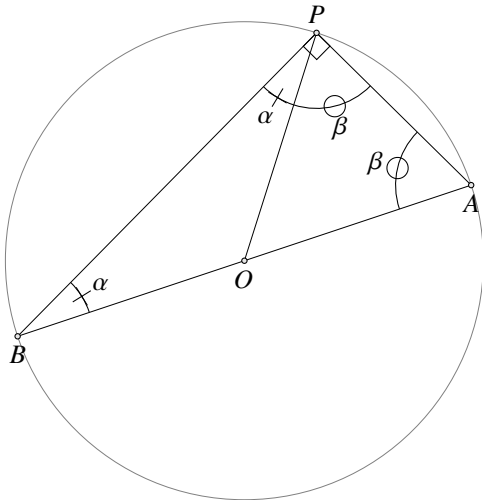
[1] Unless otherwise stated, the notation used throughout this document follows the conventions of classical French geometry and is consistent with the terminology adopted in the `tkz-elements` package.

- A *point* is denoted by a capital letter, such as A .
- The *line* determined by the points A and B is denoted by (AB) .
- The *segment* with endpoints A and B is denoted by $[AB]$. Its (unsigned) length is denoted by AB . The notation $\overline{AB}$ denotes the algebraic measure of the segment, that is, a signed length. *Remark:* Throughout this document, algebraic lengths are taken with respect to a chosen orientation on each line.
- The *ray* with origin A and passing through B is denoted by $[AB)$.
- The *vector* from A to B is denoted by $\overrightarrow{AB}$.
- The relations $(AB) \parallel (CD)$ and $(AB) \perp (CD)$ indicate that the lines (AB) and (CD) are parallel or perpendicular, respectively.
- The *angle* with vertex B and sides $[BA)$ and $[BC)$ is denoted by $\widehat{ABC}$.
- The notation ABC or $\triangle ABC$ denotes the *triangle* with vertices A , B , and C .
- The notation $C_{(O,A)}$ denotes the *circle* with centre O passing through the point A .
- The notation $C_{(O,r)}$ denotes the *circle* with centre O and radius r .

2. Collinearity and Ratios

2.1. Fundamental Theorems

Theorem 2.1 (Thales' Theorem). *An inscribed angle in a semicircle is a right angle. Thales's theorem is a special case of the inscribed angle theorem.*



Proof.

$$2\alpha + 2\beta = 180^\circ$$

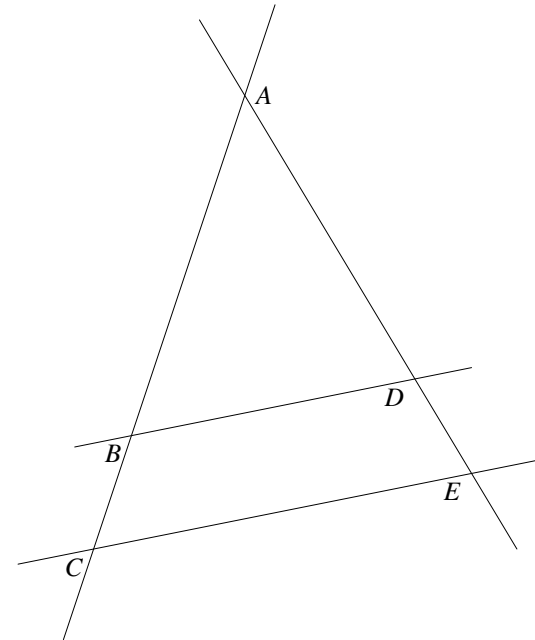
It follows that:

$$\alpha + \beta = 90^\circ$$

■

Theorem 2.2 (Intercept Theorem (Thales' Theorem)). *The intercept theorem, also known as Thales' theorem (or the basic proportionality theorem), states that if two rays with a common origin are intersected by two parallel lines, then the corresponding segments on the rays are proportional.*

Equivalently, this result expresses the proportionality of corresponding sides in similar triangles.



Proof. Let A be a point, and let B, C lie on one ray issued from A , while D, E lie on another ray issued from A . If the lines (BD) and (CE) are parallel, then

$$\frac{AB}{AC} = \frac{AD}{AE} = \frac{BD}{CE}.$$

Since $(BD) \parallel (CE)$, we have

$$\widehat{ABD} = \widehat{ACE} \quad \text{and} \quad \widehat{ADB} = \widehat{AEC}.$$

Moreover, the angle at A is common:

$$\widehat{BAD} = \widehat{CAE}.$$

Thus the triangles ABD and ACE are similar.

Therefore, the ratios of corresponding sides are equal:

$$\frac{AB}{AC} = \frac{AD}{AE} = \frac{BD}{CE}.$$

This proves the theorem. ■

2.2. Menelaus' theorem (weak version)

Theorem 2.3 (Menelaus' Theorem). *Let D , E , and F be points on the lines BC , CA , and AB respectively. Then D , E , and F are collinear if and only if*

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

Theorem 2.4. *Menelaus (weak version) If line (PR) intersecting (AB) on triangle ABC , where R is on the extension of $[AB]$, P is on (AC) , and Q on the intersection of (PR) and (BC) , then*

$$\frac{RA}{RB} \cdot \frac{QB}{QC} \cdot \frac{PC}{PA} = 1.$$

Weak version as segment lengths are used and not signed lengths.

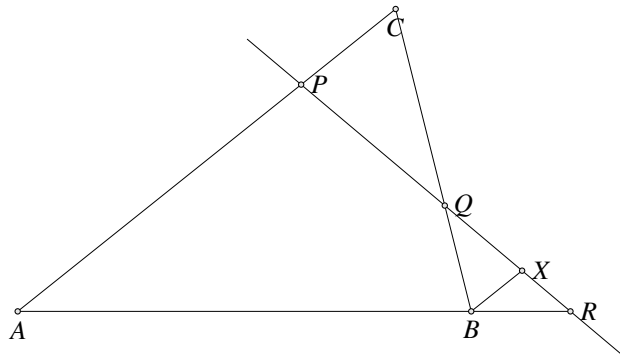
Proof. With similar triangles: draw a line parallel to (AC) from B ...

Remark It's possible to find the result by calculation because

$$\frac{RA}{RB} = \frac{|\text{length}(z.R, z.A)|}{|\text{length}(z.R, z.B)|} \text{ etc.}$$

File `menelaus.lua`:

```
init_elements()
z.A = point(0, 0)
z.B = point(6, 0)
z.C = point(5, 4)
T.ABC = triangle(z.A, z.B, z.C)
z.P = T.ABC.ca:point(0.25)
z.Q = T.ABC.bc:point(0.35)
L.PQ = line(z.P, z.Q)
z.R = intersection(L.PQ, T.ABC.ab)
z.X = intersection(T.ABC.ca:ll_from(z.B), L.PQ)
```



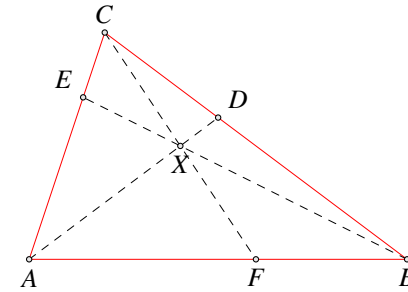
The converse of Menelaus' theorem is indeed true but we must be precise about the positioning of the points if we consider only the lengths of the segments. Suppose three points P , Q , R are on sides (or extension)

$[AB]$, $[BC]$, $[AC]$ respectively, such that one or three of P , Q , R are in the extensions of the sides. Then points P , Q , R are collinear if and only if

$$\frac{RA}{RB} \cdot \frac{QB}{QC} \cdot \frac{PC}{PA} = 1$$

Theorem 2.5 (Ceva's Theorem). *Let D , E , and F be points on the sides BC , CA , and AB respectively. Then the lines (AD) , (BE) , and (CF) are concurrent if and only if*

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$



Proof. Let ABC be a triangle, and let D , E and F be on the lines (BC) , (CA) and (AB) respectively, such that the lines (AD) , (BE) and (CF) are concurrent.

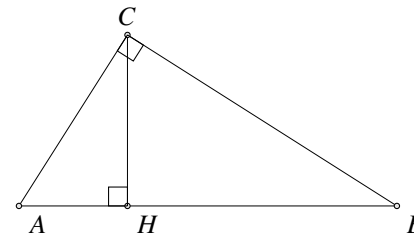
Denote the common intersection of these three lines by X . ■

3. Parallelism and Similarity

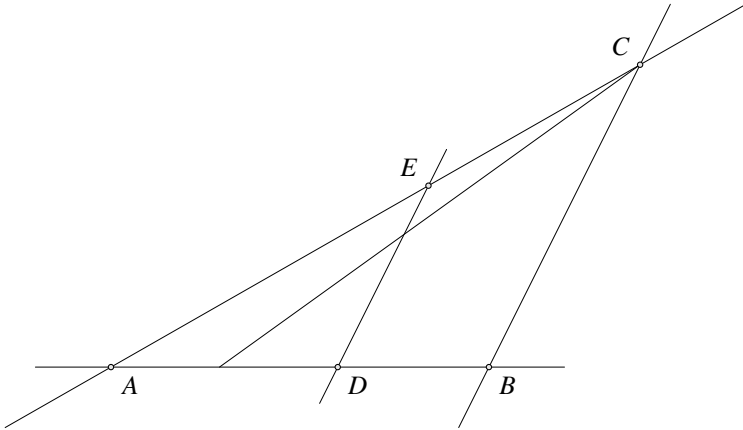
3.1. Definitions

Definition 3.1. Two triangles are *similar* if their corresponding angles are equal and their corresponding sides are proportional.

In the next figure, The right triangles ABC , ACH , and BCH are similar.



The following example is more traditional. Triangles ABC and ADE are similar.



Definition 3.2. 3.2. Definition in Affine Geometry

In an affine space P (for example, the Euclidean plane or ordinary space), given a point $O \in P$ and a nonzero scalar k , the *homothety* (or *dilation*) of center O and ratio k is the transformation f of P that leaves the point O fixed and such that, for every point $M \neq O$:

- the points O , M , and $f(M)$ are collinear;
- the directed ratio satisfies

$$\frac{\overline{Of(M)}}{\overline{OM}} = k.$$

In the most common case where the field of scalars of the affine space P is the field of real or complex numbers, a homothety of ratio k is called:

- an *enlargement* of ratio $|k|$ if $|k| > 1$;
- a *reduction* of ratio $|k|$ if $|k| < 1$.

Two particular cases should be mentioned:

- If $k = 1$, every point is invariant and the homothety is the *identity transformation*: each point is mapped onto itself;
- If $k = -1$, the homothety of ratio -1 is the *central symmetry* with center O .

A homothety preserves:

- collinearity;
- ratios of aligned segments;
- parallelism.

3.3. Basic Results

Theorem 3.3 (Midpoint Theorem). *The segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length.*

Theorem 3.4. *A homothety maps lines to parallel lines and preserves ratios of lengths.*

4. Basic Constructions

4.1. Definitions

Definition 4.1. The *perpendicular bisector* of a segment is the line perpendicular to the segment passing through its midpoint.

Definition 4.2. The *angle bisector* of an angle is the line dividing the angle into two equal angles.

Definition 4.3. The *orthogonal projection* of a point onto a line is the foot of the perpendicular from the point to the line.

4.2. Fundamental Properties

Theorem 4.4. *The perpendicular bisector of a segment is the locus of points equidistant from its endpoints.*

Theorem 4.5. *The angle bisector is the locus of points equidistant from the sides of the angle.*

Theorem 4.6. *In any triangle:*

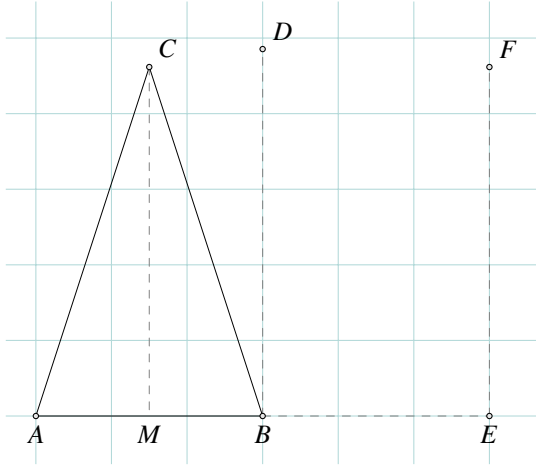
- the medians are concurrent at the centroid,
- the altitudes are concurrent at the orthocenter,
- the perpendicular bisectors are concurrent at the circumcenter,
- the angle bisectors are concurrent at the incenter.

4.3. Length transfer

Example 4.7 (Length transfer). Use of `|north` and `east` functions linked to points, to transfer lengths

File `transfer.lua`:

```
init_elements()
z.A = point(0, 0)
z.B = point(3, 0)
L.AB = line(z.A, z.B)
T.ABC = L.AB:sublime()
z.C = T.ABC.pc
z.D = z.B:north(tkz.length(z.B, z.C))
z.E = z.B:east(L.AB.length)
z.M = L.AB.mid
z.F = z.E:north(tkz.length(z.C, z.M))
```



4.4. Gold division

Here is a construction that allows segment AB to be divided into a golden ratio.

Proof. By invariance under homothety, we may assume without loss of generality that

$$A = (0, 0), \quad B = (1, 0),$$

so that $AB = 1$.

The circles with centres A and B passing respectively through B and A intersect at two points E and F . We choose the point E lying below the line (AB) . Its coordinates are

$$E = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right).$$

Let J be the midpoint of $[AB]$, hence

$$J = \left(\frac{1}{2}, 0\right).$$

Let K be the midpoint of $[JB]$, so that

$$K = \left(\frac{3}{4}, 0\right).$$

The perpendicular bisector of $[JB]$ is the vertical line of equation $x = \frac{3}{4}$. It intersects the circle with centre B and radius 1 at a point G lying above (AB) , with coordinates

$$G = \left(\frac{3}{4}, \frac{\sqrt{15}}{4}\right).$$

The line (EG) has slope

$$m = \frac{\frac{\sqrt{15}}{4} - \left(-\frac{\sqrt{3}}{2}\right)}{\frac{3}{4} - \frac{1}{2}} = \sqrt{15} + 2\sqrt{3}.$$

Its equation is therefore

$$y + \frac{\sqrt{3}}{2} = (\sqrt{15} + 2\sqrt{3}) \left(x - \frac{1}{2}\right).$$

Let C be the intersection of (EG) with the line (AB) , whose equation is $y = 0$. Substituting $y = 0$ yields

$$\frac{\sqrt{3}}{2} = (\sqrt{15} + 2\sqrt{3}) \left(x - \frac{1}{2}\right),$$

and hence

$$x - \frac{1}{2} = \frac{\frac{\sqrt{3}}{2}}{\sqrt{15} + 2\sqrt{3}} = \frac{\sqrt{5} - 2}{2}.$$

Thus

$$x = \frac{\sqrt{5} - 1}{2},$$

and consequently

$$C = \left(\frac{\sqrt{5}-1}{2}, 0\right).$$

Since $AB = 1$, it follows that

$$\frac{AC}{AB} = AC = \frac{\sqrt{5} - 1}{2}.$$

Now the golden ratio is $\varphi = \frac{1 + \sqrt{5}}{2}$, and we have

$$\frac{1}{\varphi} = \varphi - 1 = \frac{\sqrt{5} - 1}{2}.$$

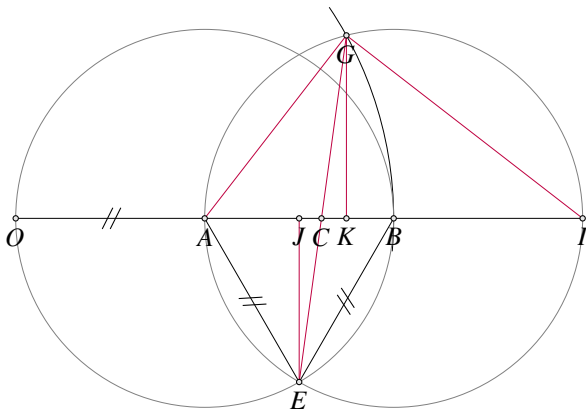
Therefore,

$$\boxed{\frac{AC}{AB} = \frac{1}{\varphi}}.$$

Hence the point C divides the segment $[AB]$ according to the golden ratio. ■

File: gold_division.lua

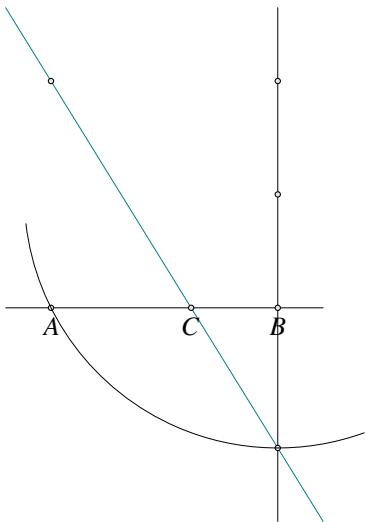
```
init_elements()
z.A = point(0, 0)
z.B = point(2.5, 0)
L.AB = line(z.A, z.B)
C.AB = circle(z.A, z.B)
C.BA = circle(z.B, z.A)
z.J = L.AB:midpoint()
L.JB = line(z.J, z.B)
z.F, z.E = intersection(C.AB, C.BA)
z.I, _ = intersection(L.AB, C.BA)
z.K = L.JB:midpoint()
L.mediator = L.JB:mediator()
z.G = intersection(L.mediator, C.BA)
L.EG = line(z.E, z.G)
z.C = intersection(L.EG, L.AB)
z.O = C.AB:antipode(z.B)
```



4.5. Gold ratio with segment

File gold_ratio.lua:

```
init_elements()
z.A = point(0, 0)
z.B = point(6, 0)
L.AB = line(z.A, z.B)
_, _, z.X, z.Y = L.AB:square():get()
L.BX = line(z.B, z.X)
z.M = L.BX.mid
C.MA = circle(z.M, z.A)
_, z.K = intersection(L.BX, C.MA)
L.AK = line(z.Y, z.K)
z.C = intersection(L.AK, L.AB)
```



Part 2. Triangle Geometry

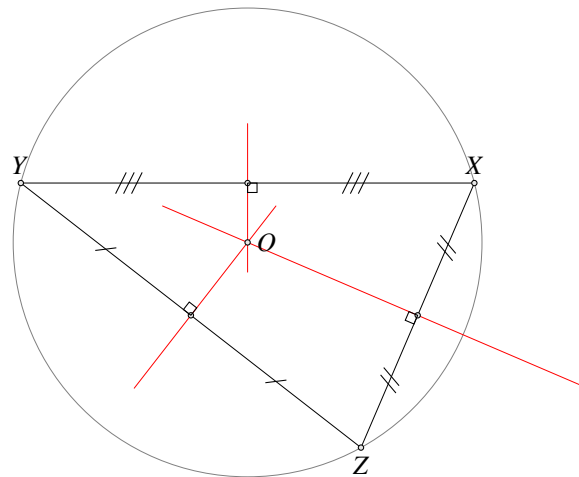
5. Basic Elements

5.1. Perpendicular Bisectors or Mediators

Theorem 5.1. *The perpendicular bisectors of the sides of a triangle intersect at a single point, called the circumcenter, denoted by O .*

Let O denote their common intersection point, the circumcenter of the triangle.

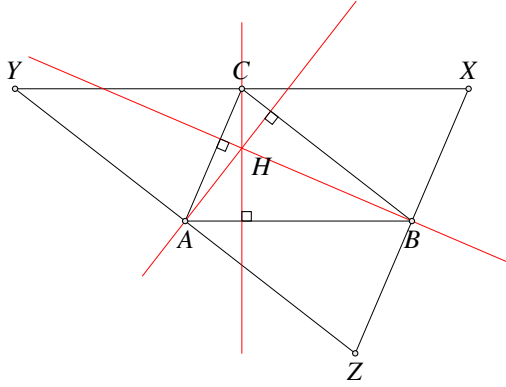
Proof. Let O be the intersection of the perpendicular bisectors of $[AB]$ and $[BC]$. Then $OA = OB$, $OB = OC$, and therefore $OA = OB = OC$. Thus, point O is the center of the circle passing through A , B , and C . ■



5.2. Altitudes

Theorem 5.2 (The altitudes are concurrent). *The altitudes of a triangle are concurrent; their common intersection point is called the orthocenter denoted by H .*

Proof. This can be seen by considering the lines through the vertices parallel to the opposite sides: the altitudes of the original triangle ABC become the perpendicular bisectors of the transformed triangle XYZ . ■



Theorem 5.3 (Inradius theorem). *In triangle ABC denote by h_a , h_b , h_c the lengths of the altitudes from the vertices A, B, C, respectively. The relation*

$$\frac{1}{r} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}$$

holds.

Proof. Let Δ be the area of triangle ABC. Since

$$\Delta = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c,$$

we have

$$a = \frac{2\Delta}{h_a}, \quad b = \frac{2\Delta}{h_b}, \quad c = \frac{2\Delta}{h_c}.$$

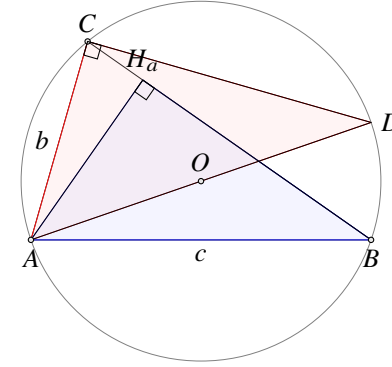
Now, using $\Delta = rs$ with $s = \frac{a+b+c}{2}$, we get

$$\frac{1}{r} = \frac{s}{\Delta} = \frac{a+b+c}{2\Delta} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}.$$

■

Theorem 5.4 (Circumradius theorem). *Denoting the altitude from one side of a triangle as h_a , the other two sides as b and c , and the triangle's circumradius (radius of the triangle's circumscribed circle) as R , the altitude is given by*

$$h_a = \frac{bc}{2R}$$



Proof. Let H_a be the foot of the altitude from A onto BC, and let D be the point on the circumcircle such that AD is a diameter ($AD = 2R$).

The right triangles ABH_a and ACD are similar, because

$$\widehat{ABH_a} = \widehat{ADC}.$$

In fact, H_a lies on BC, and the angles $\widehat{ABC}$ and $\widehat{ADC}$ subtend the same chord AC. It follows that

$$\frac{AH_a}{AC} = \frac{AB}{AD} = \frac{h_a}{b} = \frac{c}{2R}.$$

and therefore

$$h_a = \frac{bc}{2R}.$$

■

5.3. Medians

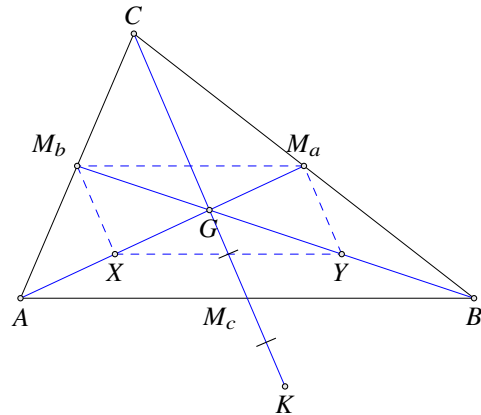
Theorem 5.5 (The medians are concurrent). *The medians of the sides of a triangle intersect at the point of concurrency, called the centroid, denoted by G.*

Proof. The medians of a triangle intersect at a single point, called the centroid, denoted by G. Let G be the point of intersection of $[AM_a]$ and $[BM_b]$, and let X and Y be the midpoints of $[GA]$ and $[GB]$.

Indeed, the quadrilateral M_bM_aYX is a parallelogram whose diagonals bisect each other. Hence the G lies two-thirds of the way from each vertex to the midpoint of the opposite side ($AG = \frac{2}{3}AM_a$). The same holds cyclically, so G is common to the three medians.

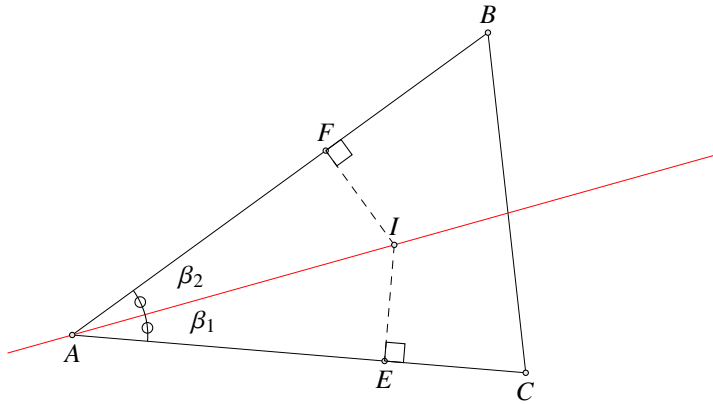
■

$$AG = \frac{2}{3}AM_a$$



5.4. Bisectors

Theorem 5.6 (The Bisector Theorem). *Any point on the bisector of an angle lies at an equal distance from the sides of that angle.*



Proof. Let I lie on the angle bisector of $\widehat{A}$, and let E and F be the feet of the perpendiculars from I to the sides.

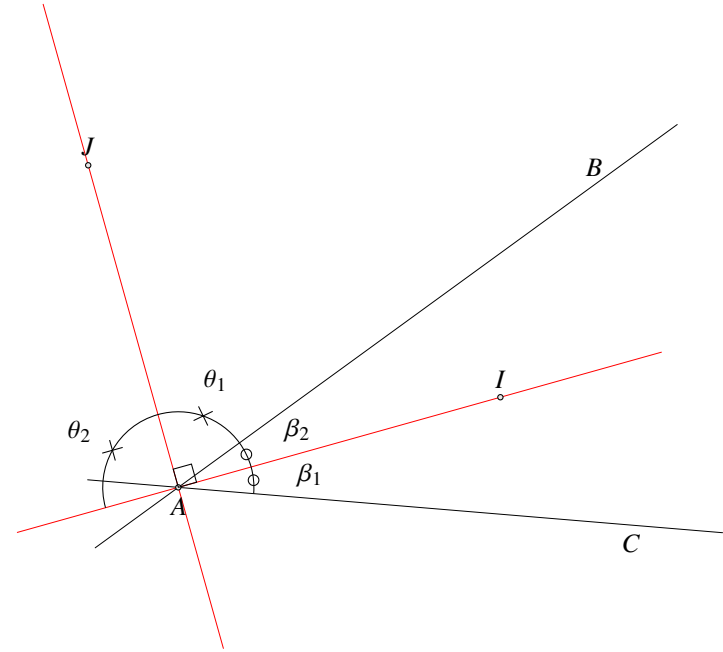
Then $\widehat{EAI} = \widehat{IAF}$ and $\widehat{AEI} = \widehat{AFI} = 90^\circ$, so triangles AEI and AFI are congruent. Hence

$$IE = IF.$$

■

Corollary 5.7. *The bisector of an angle is the locus of points equidistant from the sides of that angle.*

Theorem 5.8. *The bisectors of a pair of intersecting lines are perpendicular.*



Proof. Let β and θ be the half-angles formed by the two bisectors. Since the four adjacent angles lie on a straight line,

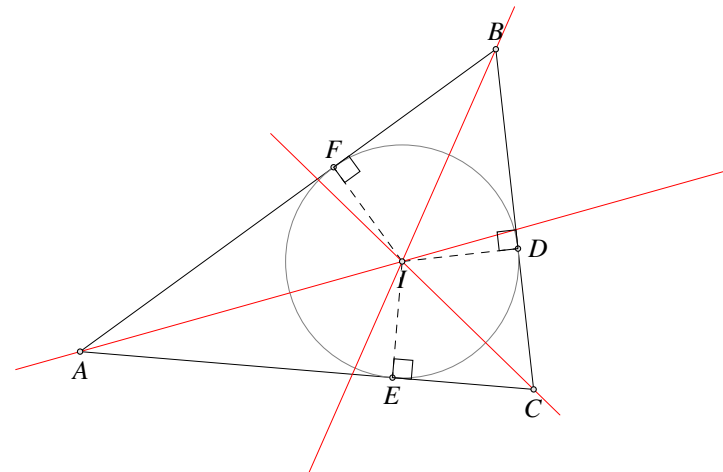
$$2\beta + 2\theta = 180^\circ.$$

Thus

$$\beta + \theta = 90^\circ,$$

which shows that the two bisectors are perpendicular. ■

Theorem 5.9. *For any triangle, the angle bisectors intersect at a single point: this point is the center of the incircle of the triangle.*



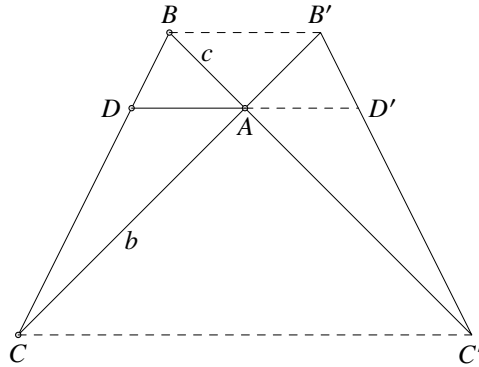
Proof. Let ABC be a triangle and let I be the intersection of the angle bisectors of $\widehat{ABC}$ and $\widehat{ACB}$. Let D , E , and F be the feet of the perpendiculars from I to BC , CA , and AB .

Since I lies on the angle bisectors, it is equidistant from the sides, so

$$ID = IE = IF.$$

Thus I is the center of a circle tangent to the three sides of the triangle. Hence I also lies on the bisector of $\widehat{BAC}$, and the three angle bisectors are concurrent. ■

Theorem 5.10. *If a ray bisects an angle $\widehat{BAC}$ of a triangle ABC , then it divides the opposite side of the triangle into segments BD and CD that are proportional to the other two sides.*



Proof. Similar triangles: BDA and BCC' , So :

$$\frac{AB}{AC'} = \frac{BD}{CD}$$

But $AC' = AC$, hence

$$\frac{AB}{AC} = \frac{BD}{CD} = \frac{c}{b}$$

■

5.5. Angle Bisector Theorem: Internal angle bisector

Theorem 5.11 (Internal angle bisector). *In a triangle, the angle bisector of any angle will divide the opposite side in the ratio of the sides containing the angle.*

Proof. In the triangle ABC , let the angle at A be bisected by the segment AD . The ray AD meets the line through C parallel to AB at a point E .

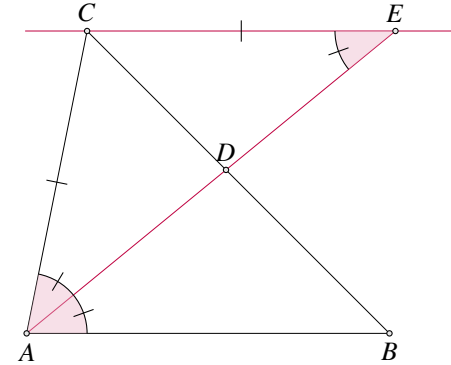
Let E be the intersection of (AD) and the line parallel to (AB) through C . $\widehat{AEC} = \widehat{BAE}$, meaning that ACE is isosceles and thus $AC = CE$. The triangles ABD and ECD are similar, implying

$$\frac{AB}{AC} = \frac{BD}{CD}.$$

With $CE = AC$, we obtain $\frac{AB}{AC} = \frac{BD}{CD}$. ■

File `internal.lua`:

```
init_elements()
z.A = point(0, 0)
z.B = point(6, 0)
z.C = point(1, 5)
T = triangle(z.A, z.B, z.C)
z.I = T.incenter
L.AI = line(z.A, z.I)
z.D = intersection(L.AI, T.bc)
L.LL = T.ab:ll_from(z.C)
L.AD = line(z.A, z.D)
z.E = intersection(L.LL, L.AD)
```



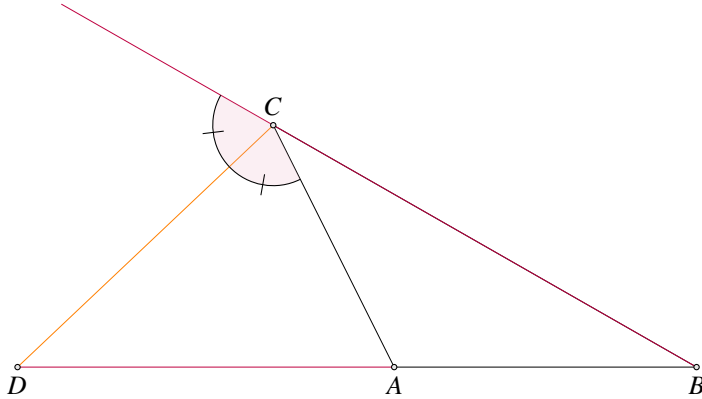
5.6. Angle Bisector Theorem: External angle bisector

Theorem 5.12 (External angle bisector). *The external bisector of an angle $(\widehat{ACB})$ of a triangle (ABC) divides the opposite side externally in the ratio of the corresponding sides containing the angle.*

If the exterior bisector intersects the line (AB) at point D then $\frac{DA}{DB} = \frac{CA}{CB}$.

File `external.lua`:

```
init_elements()
z.A = point(0, 0)
z.B = point(5, 0)
z.C = point(-2, 4)
T.ABC = triangle(z.A, z.B, z.C)
T.ext = T.ABC:excentral()
z.O = T.ABC.circumcenter
z.D = intersection(T.ext.ab, T.ABC.ab)
z.E = z.C:symmetry(z.B)
```



5.7. Specific lines of a triangle

Theorem 5.13. *The bisectors of a triangle intersect at a point called the centre of the circumscribed circle (circumcenter) of the triangle.*

Existence of the circumscribed circle

Proof. Let ABC be a triangle. The bisectors of sides $[AB]$ and $[BC]$ intersect at a point O . Then $OA = OB$ and $OB = OC$, from which we deduce that $OA = OC$, so that O belongs to the perpendicular bisector of $[AC]$. Since $OA = OB = OC$ the points A , B and C lie on a circle whose centre is O . ■

Theorem 5.14 (Intersection of altitudes). *The heights of a triangle intersect at a point called the orthocentre.*

Proof. Simply construct the triangle whose sides are parallel to the sides of the given triangle and pass through the vertices. The perpendicular bisectors of the sides of a triangle intersect. Let O be their common point, the centre of the circle circumscribing the triangle.

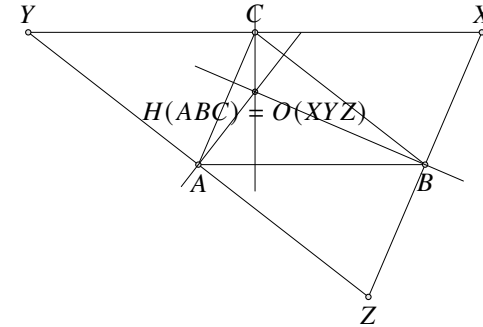
We deduce that the altitudes of a triangle are concurrent. Their common point is the orthocentre marked H . We need only consider the straight lines parallel to the sides and passing through the vertices. The altitudes of the initial triangle are the bisectors of the sides of the final triangle.

The altitudes of one triangle are the perpendicular bisectors of the other. ■

File `orthic.lua`

```
init_elements()
z.A = point(0, 0)
z.B = point(6, 0)
z.C = point(1.5, 3.5)
T.ABC = triangle(z.A, z.B, z.C)
L.YZ = T.ABC.bc:ll_from(z.A)
L.XZ = T.ABC.ca:ll_from(z.B)
L.XY = T.ABC.ab:ll_from(z.C)
z.X = intersection(L.XY, L.XZ)
```

```
z.Y = intersection(L.XY, L.YZ)
z.Z = intersection(L.XZ, L.YZ)
z.H_a, z.H_b, z.H_c = T.ABC.orthic():get()
z.H = T.ABC.orthocenter
```



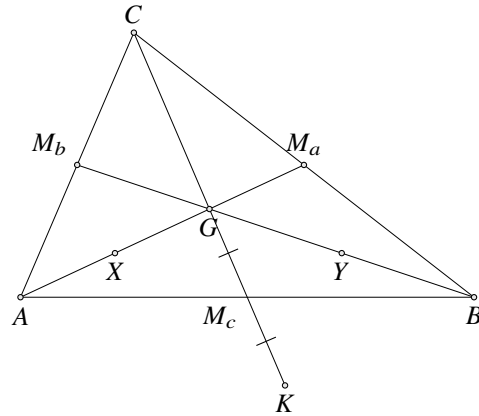
Theorem 5.15 (Intersection of medians). *The medians of a triangle intersect at a point called the isobarycentre or centre of gravity (centroid). This point, marked G , is located $2/3$ of the way along the medians.*

Proof. M_bM_aYX is a parallelogram whose diagonals intersect in the middle.

So G is two-thirds of the way between $[AM_a]$ and $[BM_b]$. The same applies to $[BM_b]$ and $[CM_c]$, so G is common to all three medians.

File `medians.lua`

```
init_elements()
z.A = point(0, 0)
z.B = point(6, 0)
z.C = point(1.5, 3.5)
T.ABC = triangle(z.A, z.B, z.C)
T.medial = T.ABC.medial()
z.G = T.ABC.centroid
z.M_a, z.M_b, z.M_c = T.medial:get()
z.X = tkz.midpoint(z.G, z.A)
z.Y = tkz.midpoint(z.G, z.B)
z.K = z.M_c:symmetry(z.G)
```



$\overrightarrow{CG} = \overrightarrow{GK}$ and $\overrightarrow{GA} + \overrightarrow{GB} = \overrightarrow{GK}$ implies that:

$$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \vec{0}$$

G est appelé isobarycentre ou centre de gravité de A, B et C .

Remarque : Pour tout point M du plan,

$$\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} = 3\overrightarrow{MG}.$$

■

6. Triangle Centers

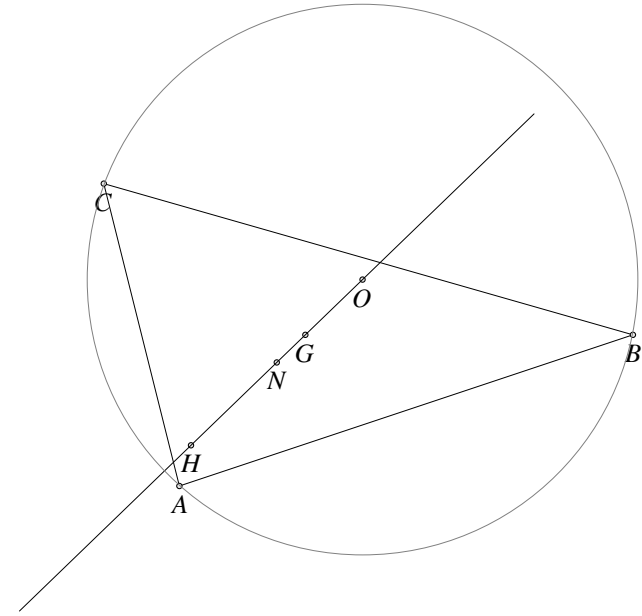
6.1. Existence of the Euler line

Suppose that the triangle $\mathcal{T}$ is not equilateral (otherwise O and G are merged). The Euler line is the line defined by these two points.

Let $\mathcal{T}$ be the triangle ABC , let O be the centre of the circle circumscribing $\mathcal{T}$, let H be the orthocentre of $\mathcal{T}$ and G its centre of gravity.

File euler_line.lua

```
init_elements()
z.A = point(0, 0)
z.B = point(6, 2)
z.C = point(-1, 4)
T.ABC = triangle(z.A, z.B, z.C)
z.O = T.ABC.circumcenter
z.H = T.ABC.orthocenter
z.G = T.ABC.centroid
z.N = T.ABC.eulercenter
```



7. Special Configurations

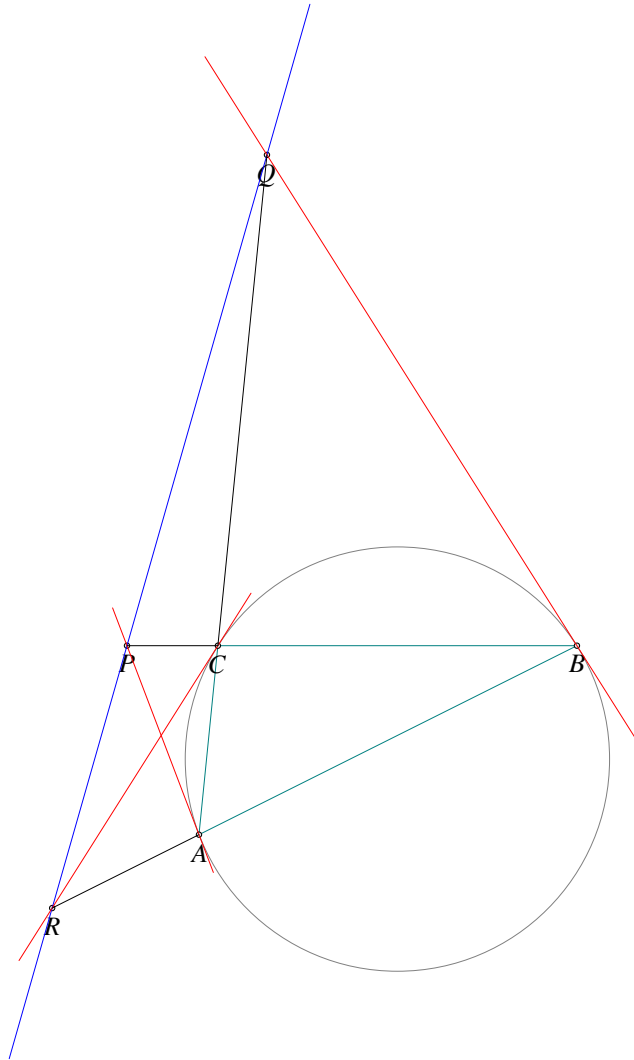
7.1. Lemoine

Theorem 7.1 (Lemoine_three tangents). *Let ABC be a triangle and let $C(ABC)$ denote its circumcircle. Let T_A , T_B , and T_C be the tangents to $C(ABC)$ at A , B , and C , respectively.*

Assume that the lines BA and CA , extended beyond A , meet T_C and T_B at P and Q , respectively. Assume also that T_A meets T_C at U and T_B at V .

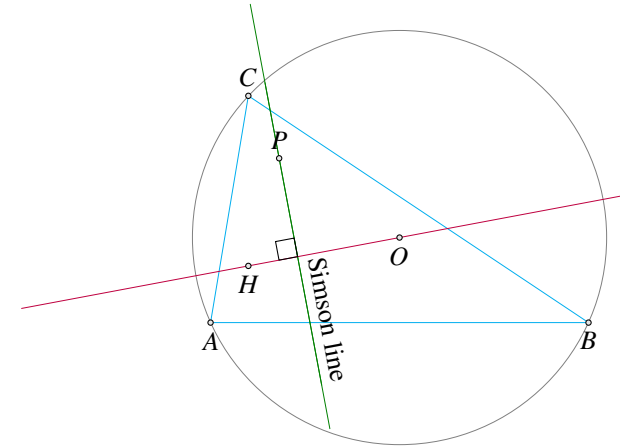
Then the three lines T_A , UV , and PQ are either concurrent or parallel.

File lemoine.lua



7.2. Orthopole and Simson line

File orthopole.lua:



7.3. Lemoine point

Definition 7.2 (Lemoine point). In Euclidean geometry, the *Lemoine point* of a triangle ABC —also known as the *symmedian point* or *Grebe point*—is the common intersection of the three symmedians, i.e., the reflections of the medians across the corresponding internal angle bisectors. Equivalently, it is the isogonal conjugate of the centroid of the triangle.

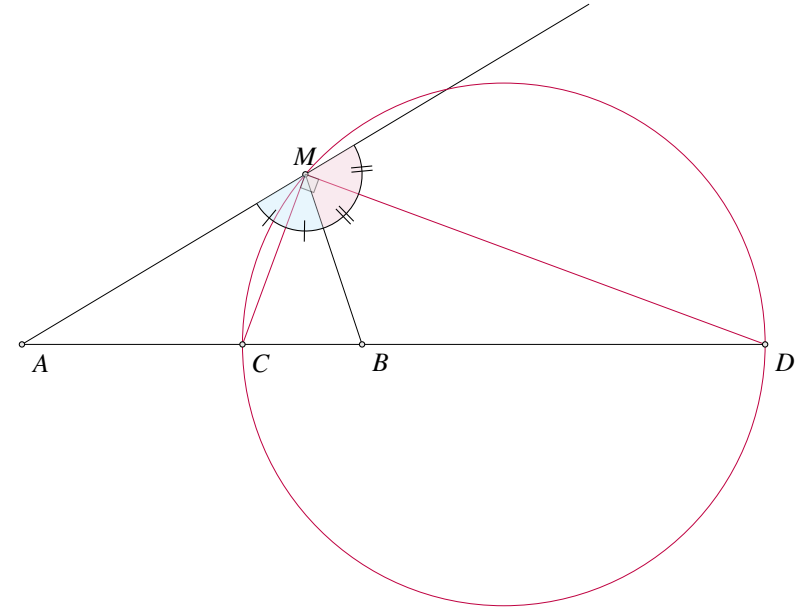
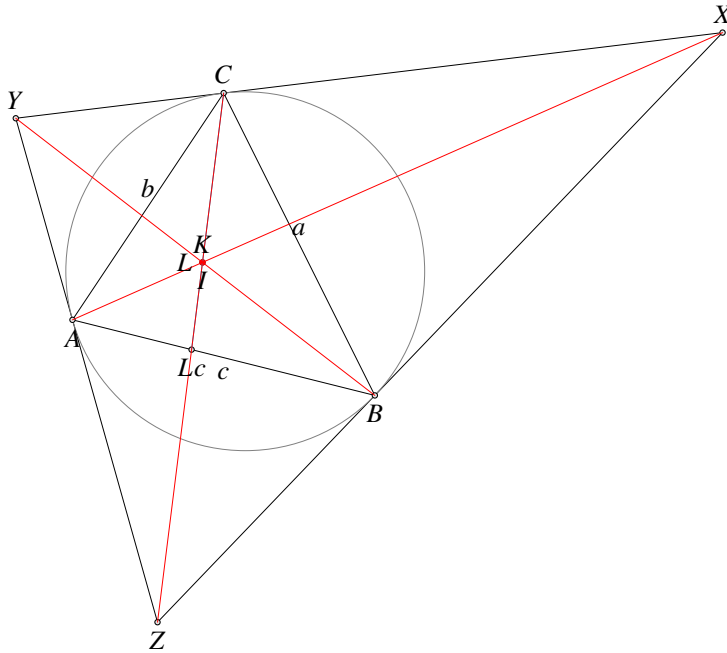
- (1) Let L be the Lemoine point, and let L_c be the foot of the symmedian issued from C . Then

$$\frac{CL}{CL_c} = \frac{a^2 + b^2}{a^2 + b^2 + c^2}.$$

- (2) In Kimberling's Encyclopedia of Triangle Centers, the Lemoine point is labelled X(6).
- (3) It is the isogonal conjugate of the centroid (center of gravity).
- (4) It is the unique point that minimizes the sum of the squares of the distances to the three sides of the triangle.
- (5) It is the barycentre of the weighted points (A, a^2) , (B, b^2) , (C, c^2) .
- (6) Its barycentric coordinates are proportional to

$$a^2 : b^2 : c^2.$$

File symmedian.lua:



Remark : The circle can be obtained with
 $|\text{local } k = \text{tkz.length}(z.M, z.A) / \text{tkz.length}(z.M, z.B)|$
 $|C.AB = T.MAB.bc:\text{apollonius}(k)|$

Part 3. Circle Geometry

8. Angles and Circles

8.1. Apollonius circle

Theorem 8.1 (Apollonius circle). *Apollonius of Perga proposed the following problem: "Find the locus of a point the ratio of whose from two fixed points is constant". Let $k \neq 1$ be the ratio such that $\frac{MA}{MB} = k$.*

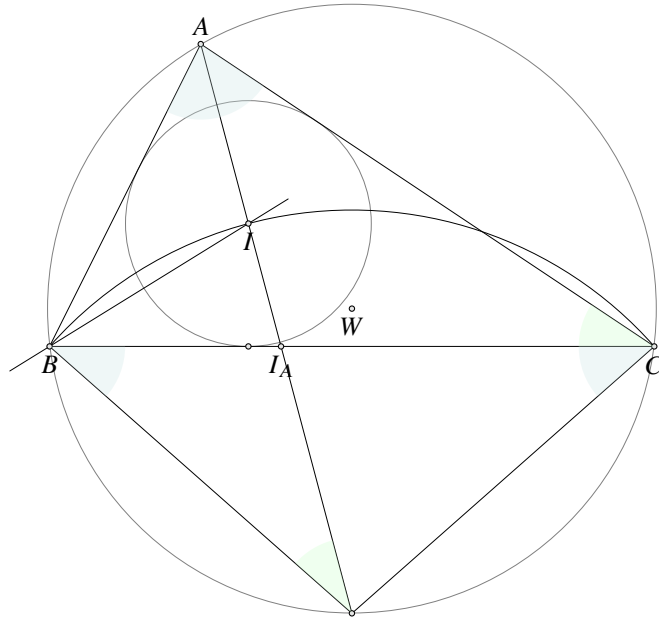
Proof. Consider the point C on the segment $[AB]$ satisfying the ratio $\frac{CA}{CB} = \frac{MA}{MB} = k$. From the "angle bisector theorem" $\widehat{AMC} = \widehat{CMB}$. Next take the point D on the line (AB) that satisfies the ratio. By the "angle bisector theorem" the line (MD) bisects the exterior angle $\widehat{EMB}$. It's easy to show that (CM) and (DM) are orthogonal in M . So M is on the circle with diameter $[CD]$. ■

8.2. South Pole

Theorem 8.2 (South Pole). *The (interior) bisector from the vertex of a triangle and the perpendicular bisector of the opposite side intersect on the circumscribed circle.*

Proof. The triangle $\widehat{BOC}$ is isosceles because $\widehat{CBO}$ and $\widehat{BCO}$ are equal ($\widehat{CBO} = \widehat{OAB} = \widehat{BCO}$). O is thus on the bisector of $[BC]$ and the arcs $\widehat{BO}$ and $\widehat{OC}$ have the same length. O is called the "south pole" of triangle ABC .

Then let us show that the triangle BIO is isosceles at O . Let $\widehat{BAC} = \beta$, $\widehat{ABC} = \gamma$ and $\widehat{BCA} = \alpha$, then $\widehat{BIO} = 180^\circ - \alpha - \frac{\beta}{2} - \frac{\gamma}{2} = \frac{\beta}{2} + \frac{\gamma}{2} = \widehat{IBO}$ so triangle BIO is isosceles at L . We deduce that $OB = OI = OC$ and B, I and C are on the circle of center O . ■



8.3. Inscribed Angles

Theorem 8.3 (Inscribed Angle Theorem). *Let O be the center of a circle, and let A , B , and C be three points on the circle. Then the inscribed angle $\widehat{ABC}$ is half the central angle $\widehat{AOC}$ subtending the same arc. In other words,*

$$\widehat{AOC} = 2\widehat{ABC}.$$

Proof. We distinguish cases according to the position of the center O relative to the angle $\widehat{ABC}$.

Case 1: O lies on one side of the angle.

Assume that O lies on the line BC . Then BC is a diameter. Since $OA = OB$, triangle AOB is isosceles, and therefore

$$\widehat{OAB} = \widehat{ABO}.$$

Hence,

$$\widehat{AOB} = 180^\circ - 2\widehat{ABO}.$$

Because B , O , and C are collinear, we also have

$$\widehat{AOB} + \widehat{AOC} = 180^\circ.$$

Thus,

$$\widehat{AOC} = 180^\circ - \widehat{AOB} = 2\widehat{ABO}.$$

Since BO and BC are the same ray, we have

$$\widehat{ABO} = \widehat{ABC},$$

and therefore

$$\widehat{AOC} = 2\widehat{ABC}.$$

Case 2: O lies inside the angle.

Let D be the point on the circle such that BD is a diameter. Then B , O , and D are collinear. By Case 1, applied to the angles $\widehat{ABD}$ and $\widehat{AOD}$, we get

$$\widehat{AOD} = 2\widehat{ABD}.$$

Applying Case 1 again to the angles $\widehat{DBC}$ and $\widehat{DOC}$, we obtain

$$\widehat{DOC} = 2\widehat{DBC}.$$

Adding these two equalities gives

$$\widehat{AOC} = \widehat{AOD} + \widehat{DOC} = 2\widehat{ABD} + 2\widehat{DBC} = 2(\widehat{ABD} + \widehat{DBC}).$$

Since BD lies inside the angle $\widehat{ABC}$,

$$\widehat{ABD} + \widehat{DBC} = \widehat{ABC}.$$

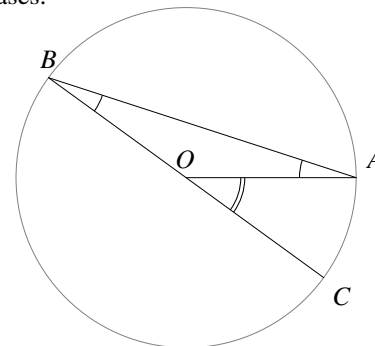
Therefore,

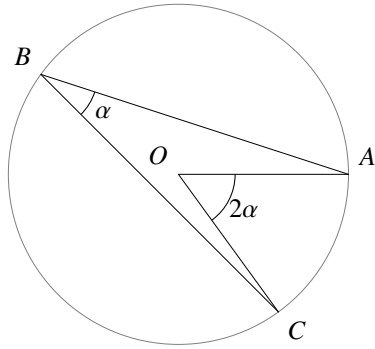
$$\widehat{AOC} = 2\widehat{ABC}.$$

This proves the theorem. ■

Theorem 8.4 (Inscribed Angle Theorem). *The inscribed angle theorem states that an angle inscribed in a circle is half of the central angle that intercepts the same arc on the circle.*

Let O be the center of a circle, and let A , B , and C be three points on that circle. Let's consider the angles $\widehat{ABC}$ and $\widehat{AOC}$. There are three cases: either point O , the center of the circle, lies on one of the sides of the angle (in which case that side is a diameter), or O lies inside the angle, or, finally, it lies outside the angle. We will consider the first two cases.





(1) O is on a side.

$$\widehat{AOB} = 180^\circ - 2 \widehat{ABO} = 180^\circ - \widehat{COA}$$

so

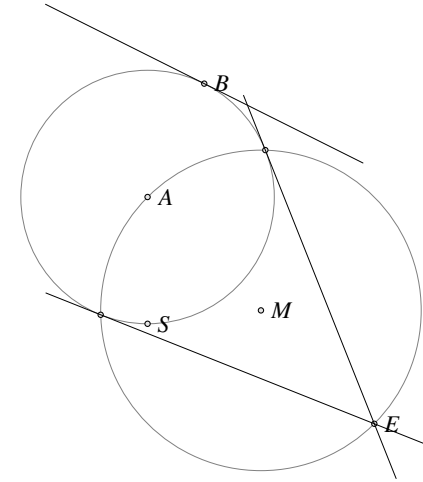
$$\widehat{COA} = 2 \widehat{ABO}$$

(2) O is inside the angle. (BO) bisects the circle at D . $[BD]$ is a diameter. It suffices to apply the previous proof to the angles $\widehat{ABD}$ and $\widehat{DBC}$.
[O is inside the angle] ■

8.4. Tangent and circle

File `tangent.lua`:

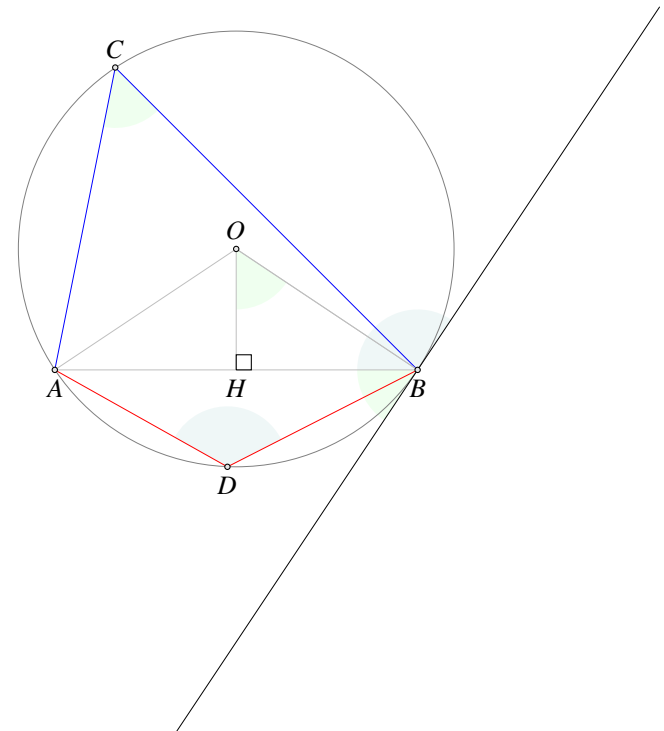
```
init_elements()
z.A = point(1, 0)
z.B = point(2, 2)
z.E = point(5, -4)
L.AE = line(z.A, z.E)
C.AB = circle(z.A, z.B)
z.S = C.AB.south
z.M = L.AE.mid
L.Ti, L.Tj = C.AB:tangent_from(z.E)
z.i = L.Ti.pb
z.j = L.Tj.pb
z.k, z.l = C.AB:tangent_at(z.B):get()
```



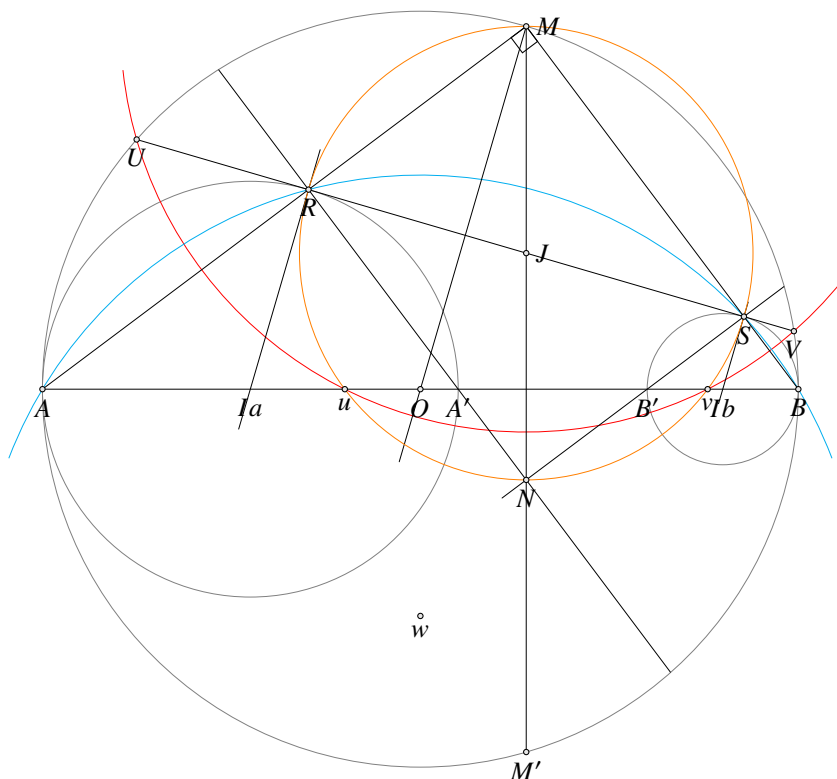
8.5. Tangent-Chord Theorem

Theorem 8.5 (Tangent-Chord Theorem). *The Tangent-Chord Theorem states that the angle formed between a chord and a tangent line to a circle is equal to the inscribed angle on the other side of the chord.*

File `tangent_chord.lua`



File outside.lua

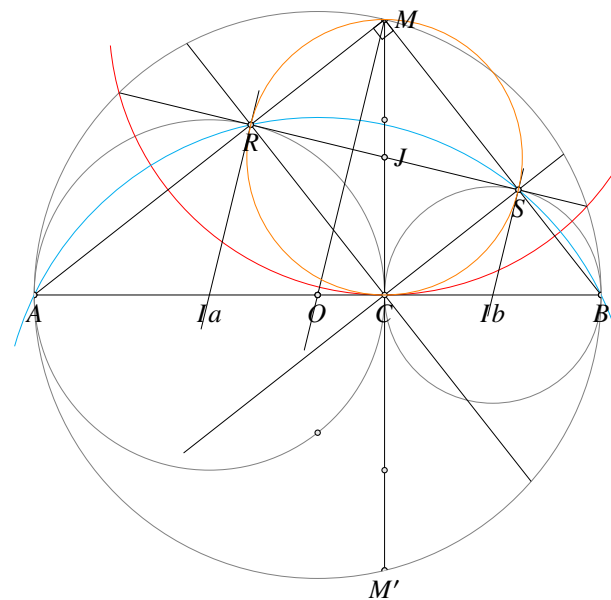


```

init_elements()

z.A = point(0, 0)
z.B = point(10, 0)
L.AB = line(z.A, z.B)
z.Ia = point(3.6, 0)
z.Ib = point(7.8, 0)
C.IaA = circle(z.Ia, z.A)
C.IbB = circle(z.Ib, z.B)
L.axis = C.IaA:radical_axis(C.IbB)
z.ra, z.rb = L.axis:get()
C.OB = L.AB:diameter()
z.O = L.AB.mid
z.M, z.Mp = intersection(L.axis, C.OB)
L.AM = line(z.A, z.M)
L.BM = line(z.B, z.M)
z.R = intersection(L.AM, C.IaA)
_, z.S = intersection(L.BM, C.IbB)
L.RS = line(z.R, z.S)
z.Ap = C.IaA:antipode(z.A)
z.Bp = C.IbB:antipode(z.B)
z.J = intersection(L.axis, L.RS)
L.RAp = line(z.R, z.Ap)
L.SBp = line(z.S, z.Bp)
z.N = intersection(L.RAp, L.SBp)
z.V, z.U = intersection(L.RS, C.OB)
C.JM = circle(z.J, z.M)
z.w = triangle(z.A, z.R, z.S).circumcenter
C.wU = circle(z.M, z.U)
C.wA = circle(z.w, z.A)

```



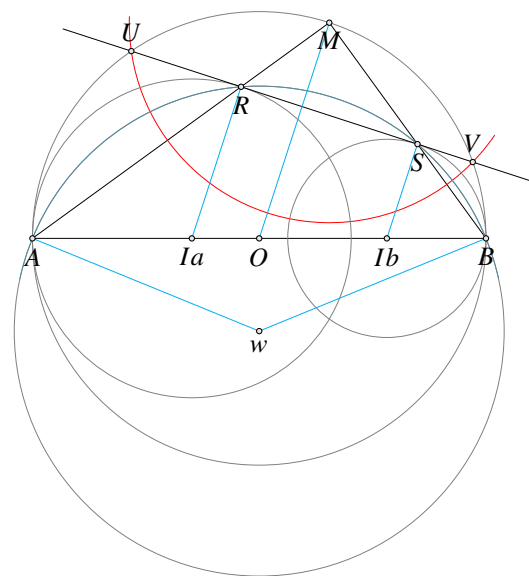
(4) Proof with inversion
File `proof_inv.lua`

Let $C(M, U)$ be the circle with center M orthogonal to the circles $C(Ia, R)$ and $C(Ib, S)$ (Note that M lies on the radical axis of the two circles). $C(M, U)$ intersects the circle $C(O, M)$ at U and V .

Let $\mathcal{I}_M$ be the inversion with center M and circle $C(M, U)/$.

The image of the circle $C(O, M)$ is the line (UV) . The images of points A and B are points R and S . We deduce that $(UV) = (RS)$.

(3) Version "Tangent outside"
File `tgt_outside.lua`



Theorem - Image of a circle passing through the ∞ - pole
Given:

$\triangle AMB$ right-angled triangle, $[AB]$ hypotenuse and diameter of circumscribed circle.

A circle of center M intercepts $[AM]$ and $[BM]$ at U and V .

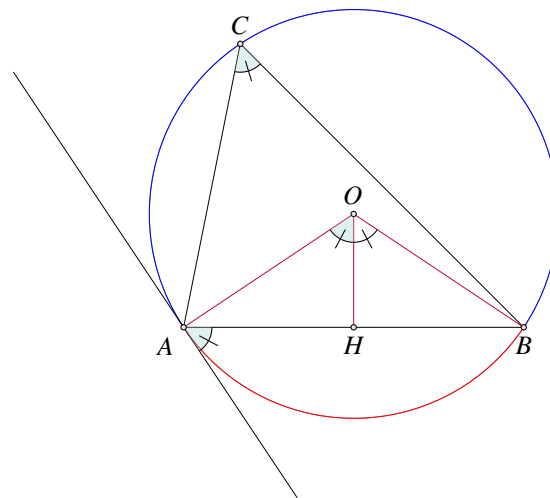
Consider the inversion with center M and circle $C(M, U)$.

The circle $C(O, M)$ passes through the inversion pole. Its image is a straight line (UV) , parallel to the tangent at M to the circle $C(O, M)$. The points U and V intersection of $C(O, M)$ with the inversion circle are invariant. The line (UV) is orthogonal to the line (MO) (parallel to the tangent at M to $C(O, M)$).

A and B have images R and S belonging to the line (UV) and to the segments $[AM]$ and $[BM]$.

The triangles RMS and AMB are similar. We deduce that the angles $\widehat{MBA}$ and $\widehat{ARS}$ are supplementary and therefore that the quadrilateral $ARSB$ is inscribable in a circle $C(w, A)$ which is the image of the line (AB) . The circle $C(w, A)$ is globally invariant and therefore orthogonal to the inversion circle $C(M, U)$.

Let's draw the parallel through R to the line (MO) . It is orthogonal at R to the line (UV) . The line (UV) is tangent to the circle $C(Ia, R)$ at R and $C(Ib, S)$ at S .



Theorem 8.6 (Alternate segment). *The Alternate Segment theorem states: An angle that lies between a tangent and a chord is equal to the angle subtended by the same chord in the alternate segment.*

Proof. $\triangle AOB$ is an isosceles triangle. $[OH]$ is the perpendicular bisector of $[AB]$ and finally $\widehat{AOB} = 2\widehat{ACB}$ then ... ■

Converse: If a line is drawn at one end of a chord of a circle in such a way that angles made with the chord are equal to alternate angles made by the chord in the segment then this line is the tangent line to the circle.

8.6. Alternate Segment Theorem

File `alternate_seg.lua`:

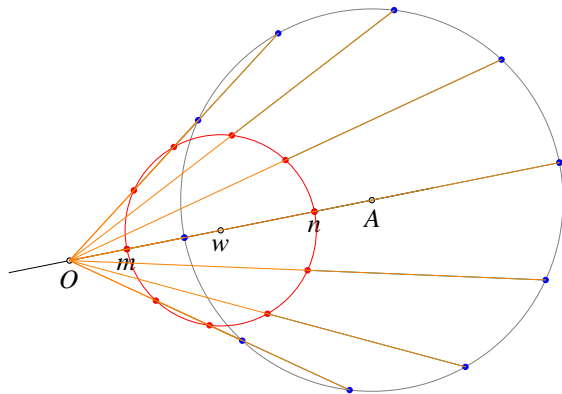
```
init_elements()
z.A = point(0, 0)
z.B = point(6, 0)
z.C = point(1, 5)
T.ABC = triangle(z.A, z.B, z.C)
L.AB = T.ABC.ab
z.O = T.ABC.circumcenter
C.OA = circle(z.O, z.A)
z.H = L.AB:projection(z.O)
L.ab = C.OA:tangent_at(z.A)
z.a, z.b = L.ab.pa, L.ab.pb
```

9. Classical Circles

9.1. Circle-Point Midpoint Theorem

Theorem 9.1 (Circle-Point Midpoint Theorem). *Taking the locus of midpoints from a fixed point to a circle of radius r results in a circle of radius $r/2$.*

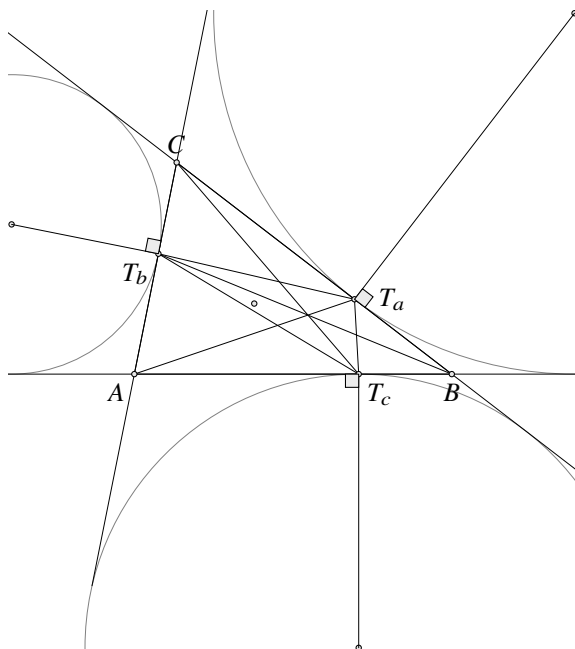
Weisstein, Eric W. "Circle-Point Midpoint Theorem." From MathWorld—A Wolfram Resource.



9.2. Excircles

Definition 9.2. Excircles An *excircle* (or *escribed circle*) of a triangle is a circle lying outside the triangle, tangent to one of its sides and tangent to the extensions of the other two sides. Every triangle has three distinct excircles, each tangent to a different side of the triangle.

File `excircles.lua`:

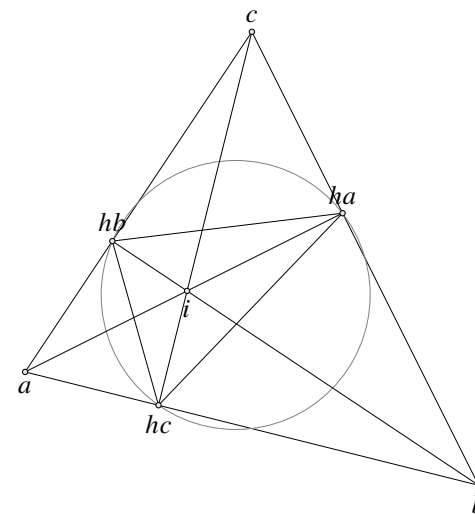


9.3. Example: Cevian circle with orthocenter

Definition 9.3. Cevian circle Given a triangle ABC and a point P , the *Cevian circle* of P is the circumcircle of the Cevian triangle of P , that is, the triangle formed by the three intersections of the cevians AP , BP , and CP with the opposite sides.

Definition 9.4. Special case If P is the orthocenter of ABC , then the Cevian circle of P coincides with the nine-point circle.

File `cevian.lua`:



10. Orthogonal Circles

10.1. Angle of two intersecting circles

Definition 10.1 (Angle of two intersecting circles). Angle at which one of the intersection points sees the segment formed by the two centers.

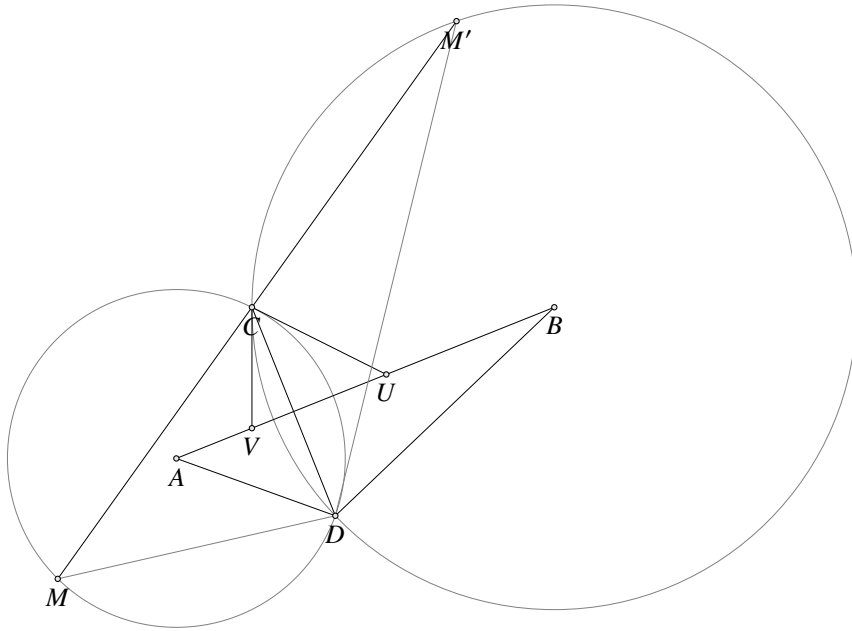
File `angle_circles.lua`

```
init_elements()
z.A = point(0, 0)
z.B = point(5, 2)
L.AB = line(z.A, z.B)
z.C = point(1, 2)
C.AC = circle(z.A, z.C)
C.BC = circle(z.B, z.C)
z.M = C.AC:point(0.45)
L.MC = line(z.M, z.C)
z.Mp = intersection(L.MC, C.BC)
_, z.D = intersection(C.AC, C.BC)
L.tAC = C.AC:tangent_at(z.C)
L.tBC = C.BC:tangent_at(z.C)
```

```

z.r, z.s = L.tAC:get()
z.u, z.v = L.tBC:get()
z.U = intersection(L.AB, L.tAC)
z.V = intersection(L.AB, L.tBC)

```



Theorem 10.2. Any secant passing through one of the points common to two circles is seen from the other common point at a constant angle equal to the angle of the two circles.

Proof. The triangles ABD and $MM'D$ are similar ... ■

Definition 10.3. The angle of two intersecting circles is therefore the angle at which a secant passing through the other common point can be seen from common point.

Theorem 10.4. The angle formed by the tangents at a point common to both circles is supplementary to the angle of the two circles.

Proof. $\widehat{UCD} = \widehat{DMC}$ and $\widehat{VCD} = \widehat{DM'C}$ ■

Theorem 10.5. Let be a tangent common to both circles at T and T' (closest to C). Let a secant parallel to this tangent pass through C . Then the segment $[TT']$ is seen from the other common point D at an angle equal to half the angle of the two circles.

```

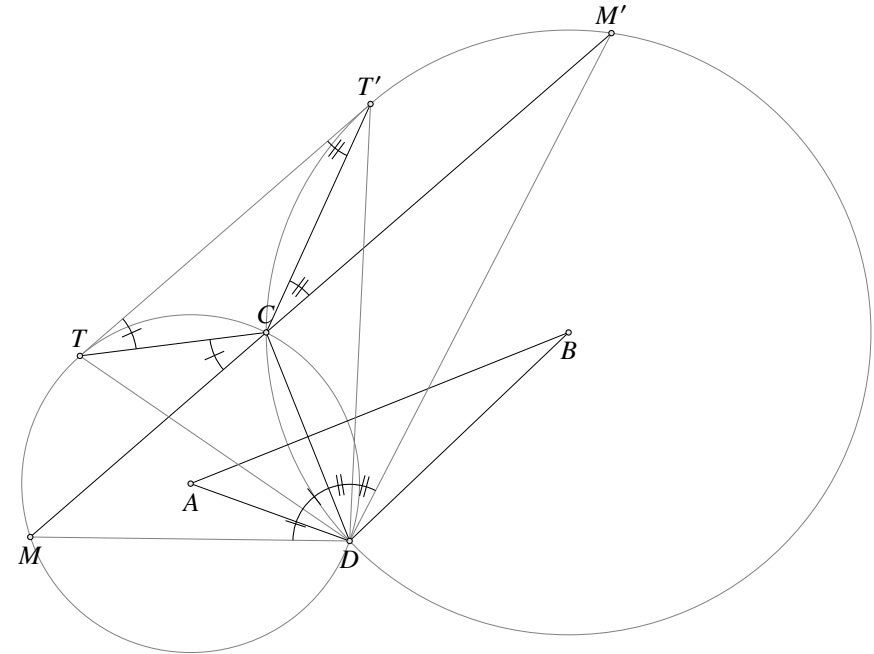
File noname.lua
init_elements ()
z.A = point(0, 0)
z.B = point(5, 2)
L.AB = line(z.A, z.B)
z.C = point(1, 2)

```

```

C.AC = circle(z.A, z.C)
C.BC = circle(z.B, z.C)
L.TTp = C.AC:common_tangent(C.BC)
z.T, z.Tp = L.TTp:get()
z.M = C.AC:point(0.45)
L.MC = line(z.M, z.C)
z.Mp = intersection(L.MC, C.BC)
L.mm = L.TTp:ll_from(z.C)
_, z.M = intersection(L.mm, C.AC)
z.Mp = intersection(L.mm, C.BC)
_, z.D = intersection(C.AC, C.BC)

```



Proof. A few angle relationships show that T and T' are the midpoints of the arcs $\widehat{MC}$ and $\widehat{M'C}$. (DT) and (DT') are bisectors of $\widehat{MDC}$ and $\widehat{CDM'}$.

So $\widehat{MDM'} = 2\widehat{TDT'}$ ■

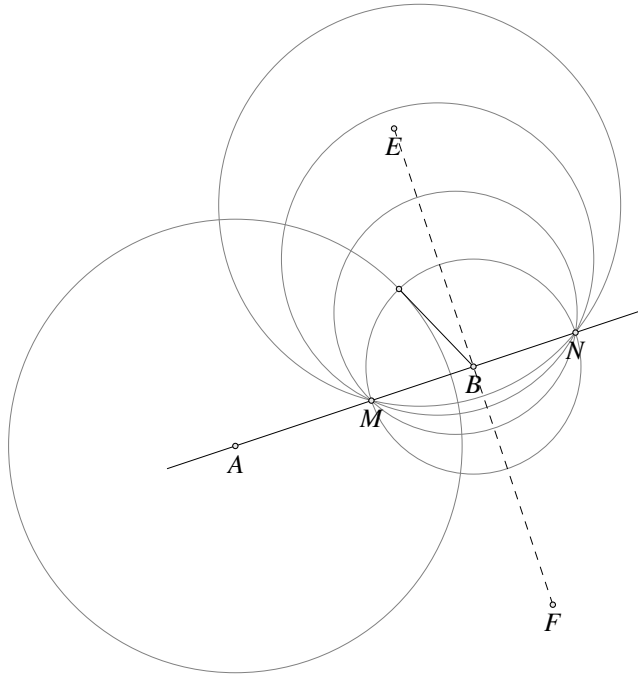
Theorem 10.6 (Orthogonality condition). For two circles to be orthogonal, it is necessary and sufficient for a secant passing through one of their common points to be seen from the other common point at a right angle.

File orthogonality.lua

```

init_elements()
z.A = point(0, 0)
z.B = point(4, 2)
L.AB = line(z.A, z.B)
z.a = point(1, 2)

```

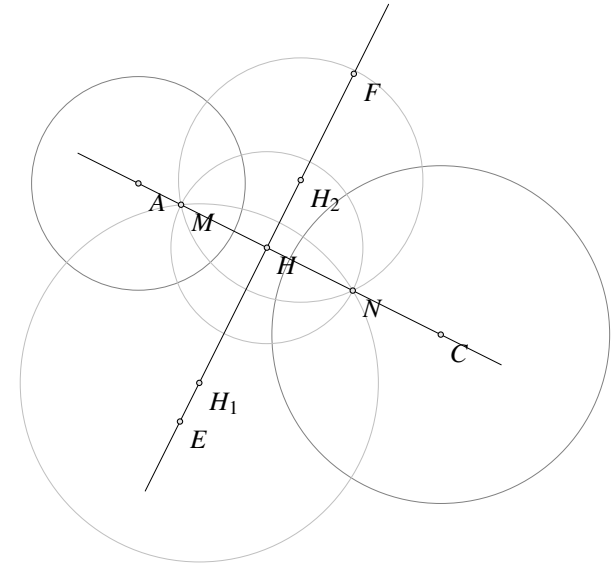
$$\frac{NB}{NA} = \frac{MB}{MA} \text{ The division is harmonic } : (A, B; M, N) = -1$$

All circles whose center lies on the radical axis (EF) and passing through N are orthogonal to the circle $C(A)$.
see the pencils

10.3. Circle orthogonal to two given circles

Theorem 10.10. *Circle orthogonal to two given circles* If a circle is orthogonal to two given circles, its center lies on the radical axis of those two circles.

File ortho_two.lua:



10.4. Orthogonal Circles – Secant Criterion

For two circles to be orthogonal, it is necessary and sufficient for a secant passing through one of their common points to be seen from the other common point at a right angle.

Theorem 10.11. *Orthogonal Circles – Secant Criterion* Let two circles intersect at points P and Q . Then the circles are orthogonal if and only if any secant through P is seen from Q at a right angle; that is, for every point X on any secant through P , one has

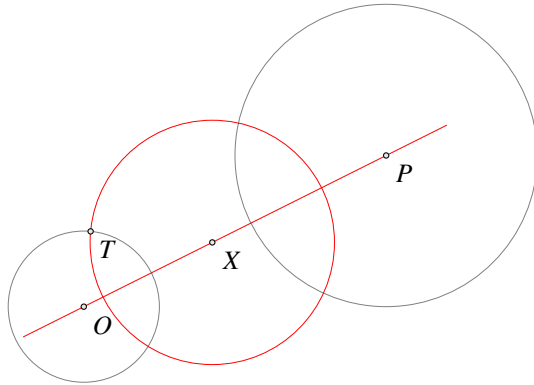
$$\angle XQP = 90^\circ.$$

File ortho_circles.lua

- Case of two circles. The functions below can be used to obtain the center (located on the center axis and the radical axis) of a circle orthogonal to the two given circles.

File radical_two.lua:

```
init_elements()
z.O = point(0, 0)
z.x = point(1, 0)
z.z = point(4, 0)
z.P = point(4, 2)
C.Ox = circle(z.O, z.x)
C.Pz = circle(z.P, z.z)
z.X = C.Ox:radical_center(C.Pz)
C.X = C.Ox:radical_circle(C.Pz)
z.w, z.T = C.X:get()
```

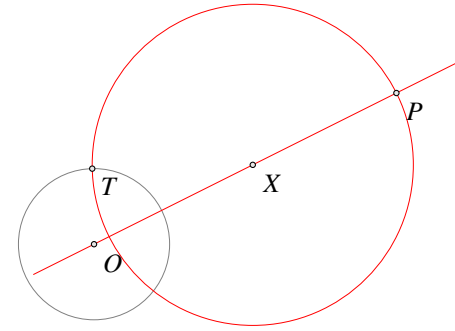


- Case of one circle and one point

The functions used for the next case are valid. We use a circle_point [C.P = circle : new (z.P,z.P)].

File radical_one.lua:

```
init_elements()
z.O = point(0, 0)
z.x = point(1, 0)
z.P = point(4, 2)
C.Ox = circle(z.O, z.x)
C.P = circle(z.P, z.P)
z.X = C.Ox:radical_center(C.P)
C.X = C.Ox:radical_circle(C.P)
z.w, z.T = C.X:get()
```

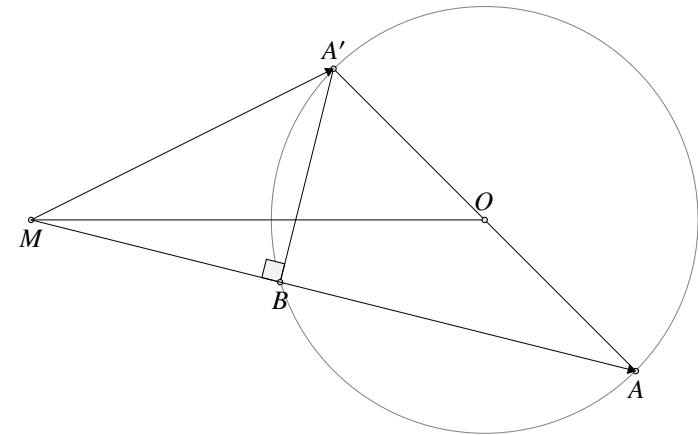


11.1. Power of a point with respect to a circle

Definition 11.2. Power of a point with respect to a circle When a variable secant from a fixed point M intersects a given circle C at A and B , the product $\overline{MA} \times \overline{MB}$ is a constant number called the power of point M with respect to this circle.

These points are of course merged if the line is tangent to the circle. The number k is positive if the point M is outside the circle, null if it belongs to it and negative if it belongs to the open disk defined by the circle. We can easily verify that $k = d^2 - R^2$ where d is the distance from point M to the center of the circle and R is the radius of the circle. k est aussi noté $-(M)$. We therefore have $-(M) = \overline{MA} \times \overline{MB} = d^2 - R^2$.

File power.lua:



11.2. Properties of the power

- (1) The power of a point M with respect to a circle C of diameter AB is equal to the scalar product $\overrightarrow{MA} \cdot \overrightarrow{MB}$

$$\text{Proof. } \overrightarrow{MA} \cdot \overrightarrow{MB} = (\overrightarrow{MO} + \overrightarrow{OA}) \cdot (\overrightarrow{MO} - \overrightarrow{OA}) = \overrightarrow{MO}^2 - \overrightarrow{OA}^2 = d^2 - R^2 \quad \blacksquare$$

$$-(M) = \overrightarrow{MA} \cdot \overrightarrow{MB}.$$

- (2) If we draw two secants (MAB) and (MCD) to the circle – we obtain :

$$-(M) = \overrightarrow{MA} \cdot \overrightarrow{MB} = \overrightarrow{MC} \cdot \overrightarrow{MD}$$

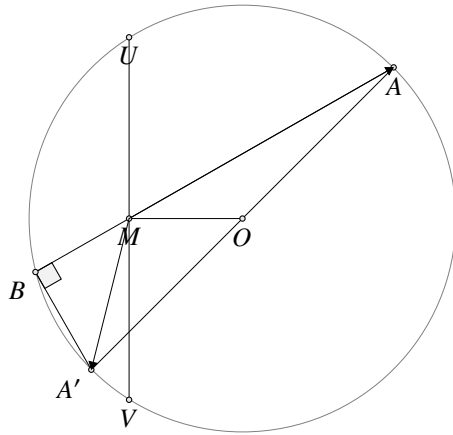
If the point M is outside the circle, denoting by MT a tangent :

$$-(M) = MT^2 = \overline{MA} \times \overline{MB}$$

If the point M is interior , designating by $[UV]$ the perpendicular chord to (OM) :

$$-(M) = -\overline{MU}^2$$

File power_prop.lua:



- (3) For the power $-(M)$ to be positive, zero or negative, it is necessary and sufficient that d is greater than, equal to or less than R , so that M is outside, on the circle or inside the circle -. The minimum of $-(M)$ is reached when M is in O : $-(M) = -R^2$.
- (4) If the sides AB and CD of the quadrangle $ABCD$ intersect at a point M such that $\overrightarrow{MA} \cdot \overrightarrow{MB} = \overrightarrow{MC} \cdot \overrightarrow{MD}$, this quadrangle is inscribable.
- (5) If the point M on side AB of triangle ABC is such that $MC^2 = MA \times MB$ the line (MC) is tangent at C to the circle ABC .

11.3. Powers of a point with respect to two circles

Theorem 11.3 (Powers of a point with respect to two circles). *The difference of the powers of a point with respect to two circles is, in absolute value, equal to the double product of the distance of their centers by the distance of this point to their radical axis.*

(Refer to 11.6)

11.4. Pencil of circles

Definition 11.4. Pencil of circles A pencil of circles or coaxial system is any family of circles which have the same radical axis Δ .

Any two circles in the plane have a common radical axis, which is the line consisting of all the points that have the same power with respect to the two circles. This is the case, for example, for circles passing through two fixed points A and B or of circles orthogonal to two fixed circles. We will show that a pencil is defined when we know a circle of this pencil and the common radical axis Δ and consequently when we know two circles of the pencil. The centers of the circles of a pencil are aligned on a perpendicular to the radical axis Δ of the pencil called "line of centers of the pencil".

11.5. Fixed point pencil

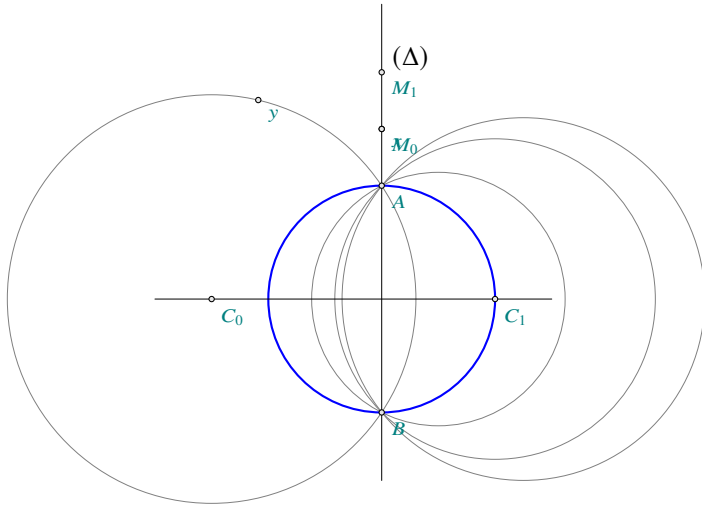
The fixed points A and B are the "base points" of the pencil.

The circles of the beam are the circles passing through the base points A and B .

The pencil has a smallest circle, having as diameter the segment $[AB]$.

File pencil1.lua:

```
init_elements()
z.A = point(0, 2)
z.B = point(0, -2)
z.C_0 = point(-3, 0)
z.C_1 = point(2, 0)
z.C_3 = point(2.5, 0)
z.C_5 = point(1, 0)
L.BA = line(z.B, z.A)
z.M_0 = L.BA:point(1.25)
z.M_1 = L.BA:point(1.5)
C.C0A = circle(z.C_0, z.A)
z.x, z.y = C.C0A:orthogonal_from(z.M_0):get()
z.xp, z.yp = C.C0A:orthogonal_from(z.M_1):get()
z.O = L.BA.mid
```



11.6. Radical axis

Definition 11.5. Radical axis The geometric set of points that have the same power with respect to two given circles is a line perpendicular to the line of the centers, called radical axis of the two circles. If the circles have two points in common, the radical axis is the common secant line of the circles.

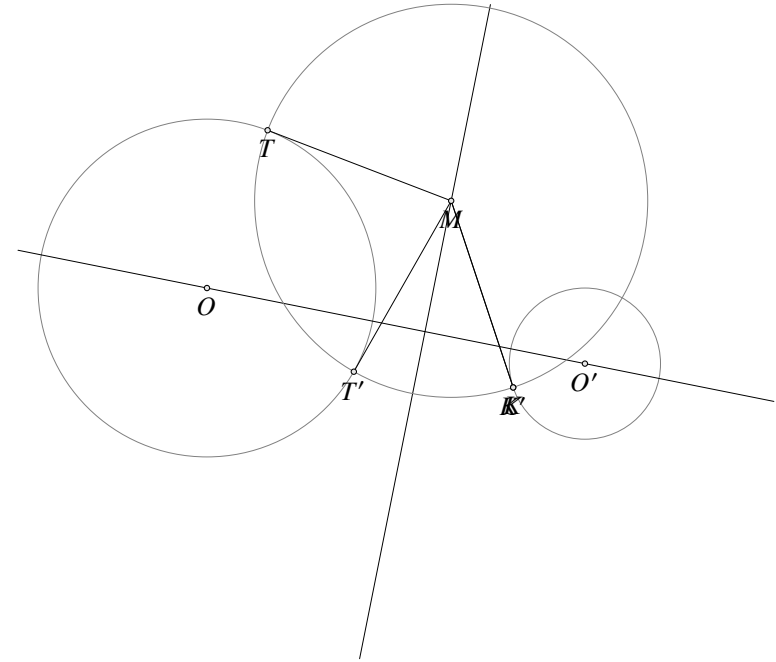
For the point M to have the same power with respect to the two circles $\Gamma = O(R)$ and $\Gamma' = O'(R')$, it is necessary and sufficient that :

$$MO^2 - R^2 = MO'^2 - R'^2 \text{ either } MO^2 - MO'^2 = R^2 - R'^2$$

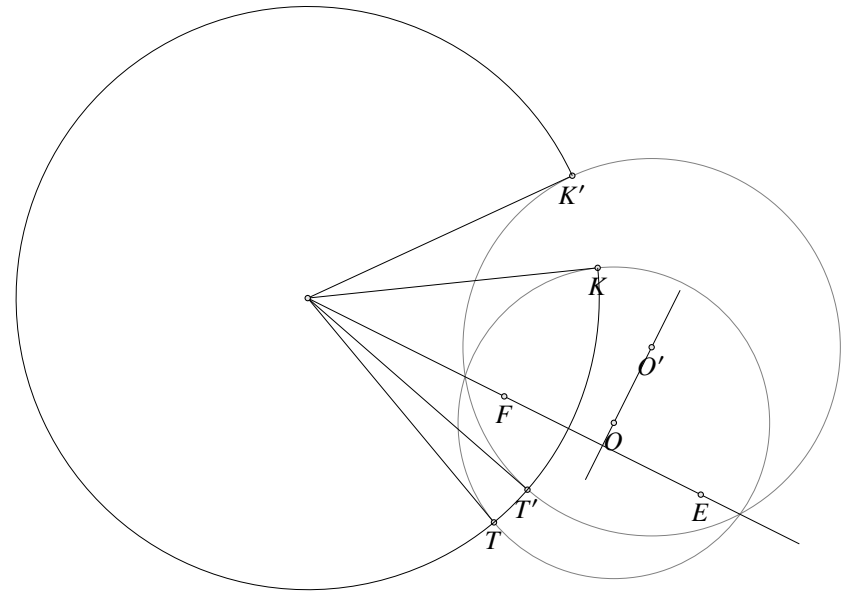
Denoting by I the middle of OO' , the set of points M is therefore a line OO' perpendicular to the line OO' at the point H defined by the relation:

$$2\overline{OO'} \cdot \overline{IH} = R^2 - R'^2$$

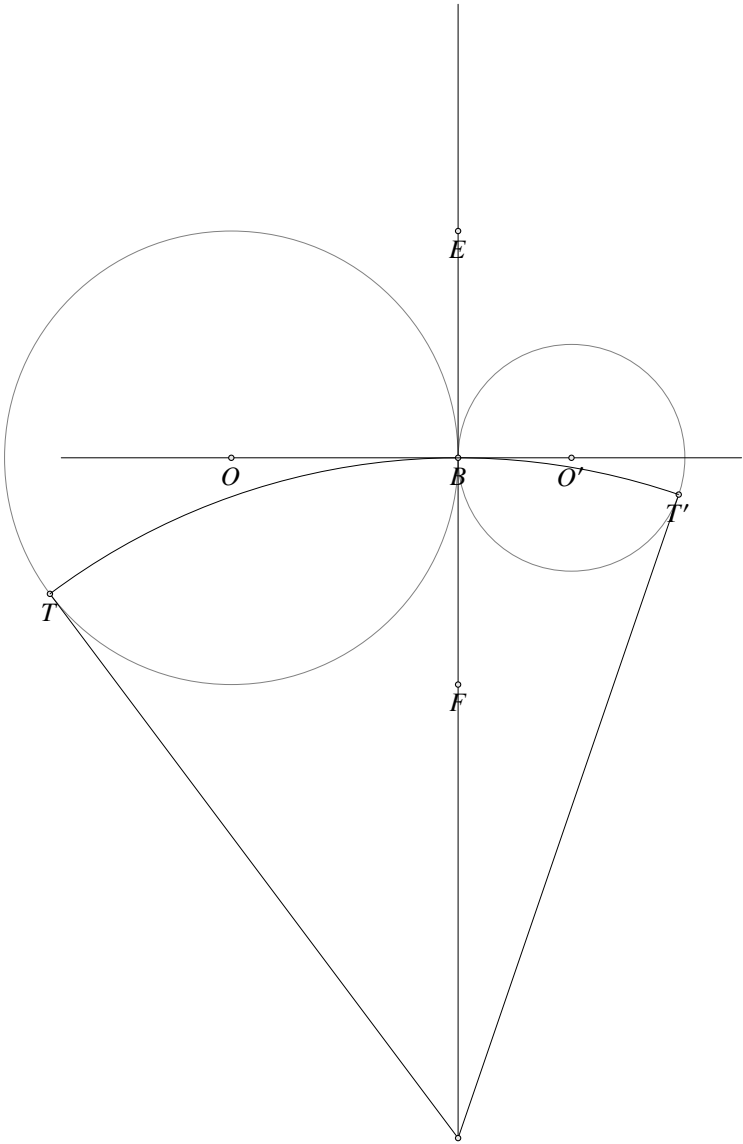
File radical_axis.lua:



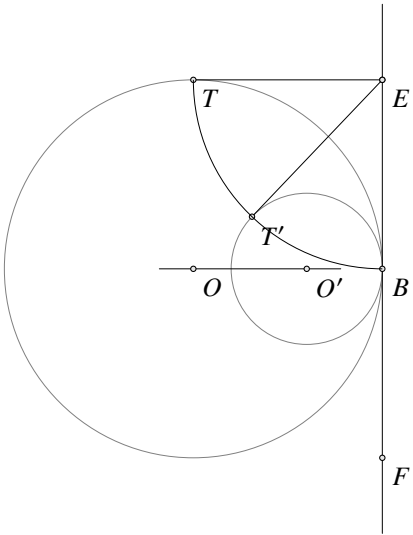
File radical_axis2.lua:



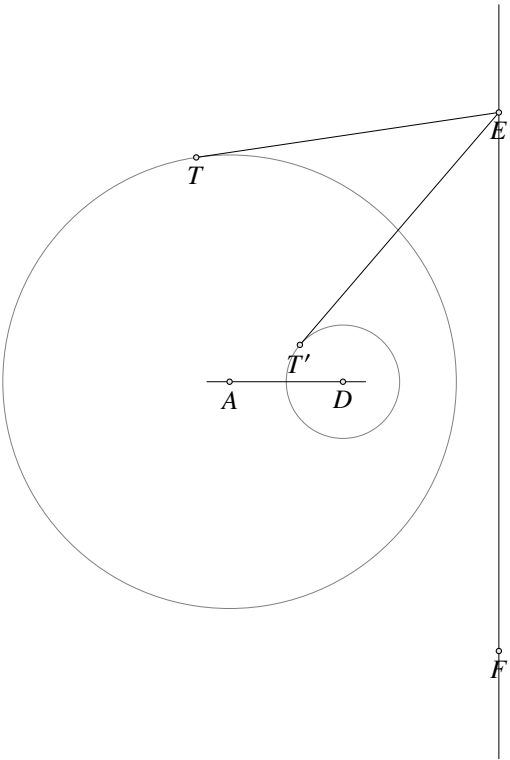
File radical_axis3.lua:



File radical_axis4.lua:



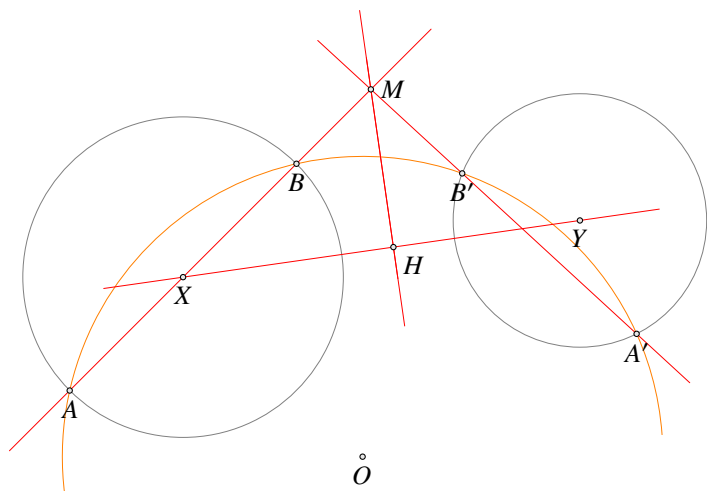
File radical_axis5.lua:



11.7. Construction of the radical axis for two non-intersecting circles

The last construction shows that in a general case the radical axes of three circles intersect at a single point. This construction allows us to construct the radical axis of two non-intersecting circles. Let us draw an auxiliary circle that intersects the first circle at A and B and the second at A' and B' . The lines (AB) and $(A'B')$ intersect at a point M which will belong to the radical axis. This one being perpendicular to the line of centers, it remains to construct the orthogonal projection of M on the line of centers.

File radical build.lua:



The part of the radical axis, outside the two circles, is the place of the points from which equal tangents can be drawn.

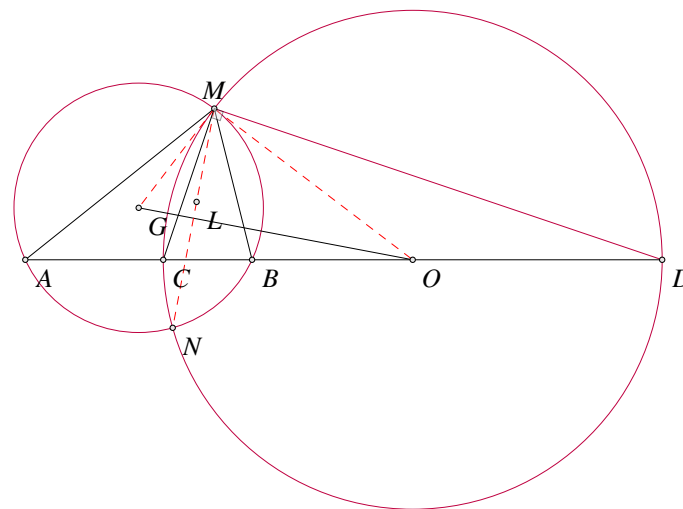
Indeed, for $MT = MT'$ it is necessary and sufficient that M has the same positive power with respect to the circles O and O' . It follows that the radical axis of two circles passes through the middle of any line common to these two circles. When two circles are external, the middles of the 4 common tangents are aligned on the radical axis.

11.8. Radical axis of circumcircle and Apollonius circle

File radical.lua:

```
init_elements()
z.A = point(0, 0)
z.B = point(6, 0)
z.M = point(5, 4)
T.AMB = triangle(z.A, z.M, z.B)
L.AB = T.AMB.ca
z.I = T.AMB.incenter
L.MI = line(z.M, z.I)
z.C = intersection(L.AB, L.MI)
L.MJ = L.MI.ortho_from(z.M)
z.D = intersection(L.AB, L.MJ)
```

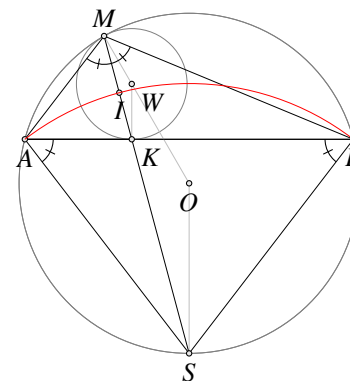
```
L.CD = line(z.C, z.D)
z.O = L.CD.mid
z.G = T.AMB.circumcenter
C.GA = circle(z.G, z.A)
C.OC = circle(z.O, z.C)
_, z.N = intersection(C.GA, C.OC)
z.L = T.AMB:lemoine_point()
```



11.9. Circle in circular segment

There is a tangent circle C to the main circle at M and I . The points M , I and S (south pole) are aligned. The circle $C(S, A)$ is orthogonal to C . The tangents at their points of contact pass through S .

File segmentaire.lua



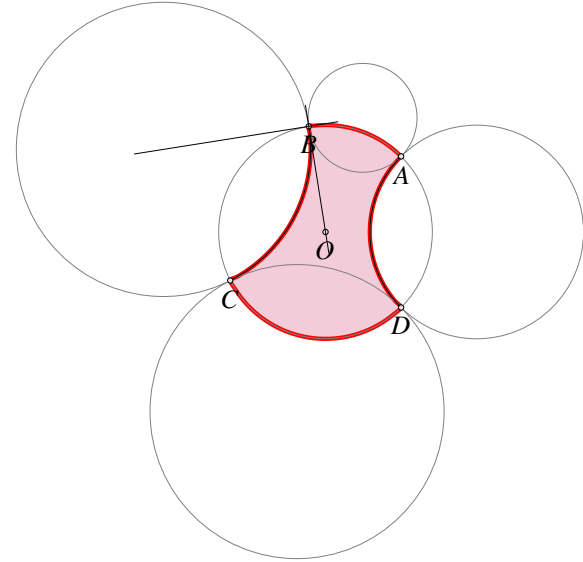
Proof. The fact that MI passes through the midarc follows by taking the homothety at M mapping $C_{W,M}$ to $C_{S,A}$: it should map the “south pole” of $C_{W,M}$ to that of $C_{S,A}$.

ergo it maps S to I . Meanwhile, $SI \times SM = SA^2$ follows from triangle MAI similar to SBI .

Theorem "South Pole" : The bisector at M passes through the south pole S .

$MWI = MOS$ so $IMW = SMO$ and M, I and S are aligned.

The inversion of center S and circle $C_{S,A}$ transforms the circle $C_{O,M}$ into a straight line (A, B) . The image of M is I , which is probably sufficient to justify the alignment. The circle $C_{W,M}$ is globally invariant and therefore orthogonal to $C_{S,A}$. The tangents at the points of intersection pass through S .



11.10. Circle and path

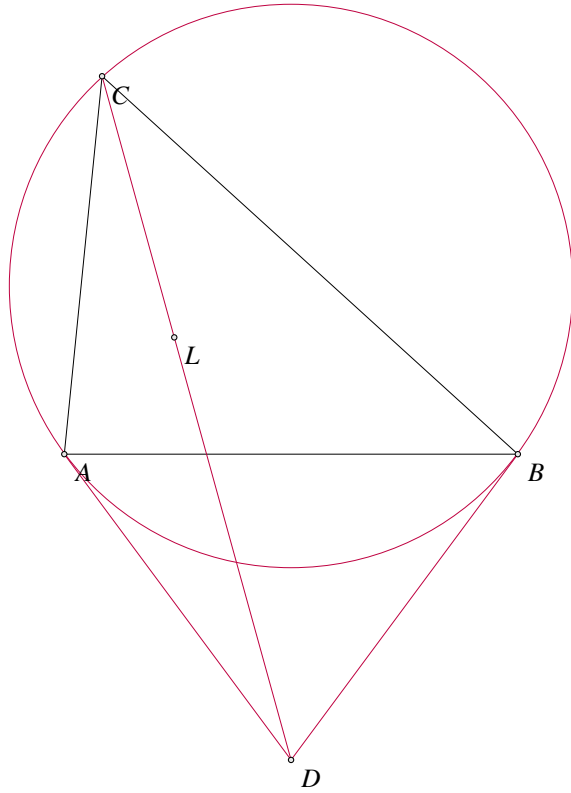
File `path_circle.lua`:

```
init_elements()
z.O = point(0, 0)
z.A = point(1, 1)
C.OA = circle(z.O, z.A)
z.B = C.OA:point(0.15)
z.C = C.OA:point(0.45)
z.D = C.OA:point(0.75)
C.RBC = C.OA:orthogonal_through(z.B, z.C)
z.R = C.RBC.center
C.SDA = C.OA:orthogonal_through(z.D, z.A)
z.S = C.SDA.center
C.TAB = C.OA:orthogonal_through(z.B, z.A)
z.T = C.TAB.center
C.UCD = C.OA:orthogonal_through(z.C, z.D)
z.U = C.UCD.center
PA.AB = C.OA:path(z.A, z.B, 20)
PA.CD = C.OA:path(z.C, z.D, 20)
PA.CB = C.RBC:path(z.C, z.B, 20)
PA.AD = C.SDA:path(z.A, z.D, 20)
PA.zone = PA.AB - PA.CB + PA.CD - PA.AD
```

11.11. Symmedian and tangents

File `symm_tgtts.lua`:

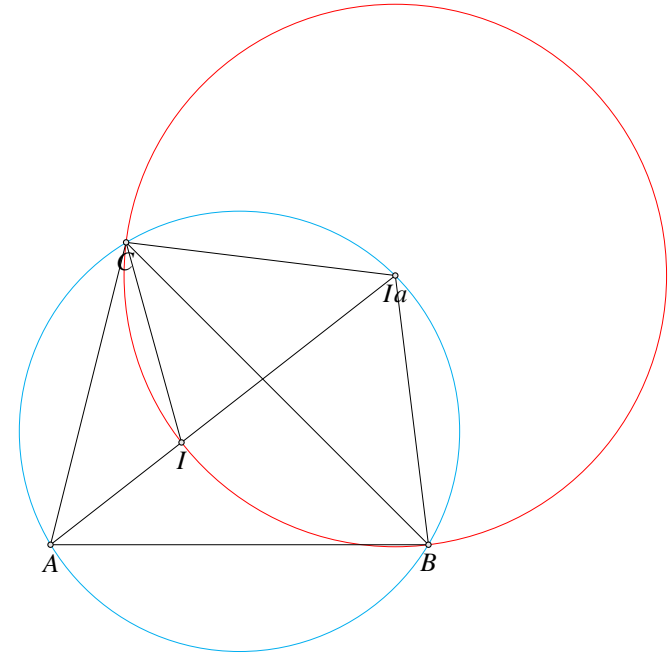
```
init_elements()
z.A = point(0, 0)
z.B = point(6, 0)
z.C = point(.5, 5)
T.ABC = triangle(z.A, z.B, z.C)
C.cir = T.ABC:circum_circle()
z.O = C.cir.center
L.A = C.cir:tangent_at(z.A)
L.B = C.cir:tangent_at(z.B)
z.D = intersection(L.A, L.B)
z.L = T.ABC:lemoine_point()
```



11.12. Circle through incircle

File incircle.lua

```
init_elements()
z.A = point(0, 0)
z.B = point(5, 0)
z.C = point(1, 4)
T.ABC = triangle(z.A, z.B, z.C)
z.I = T.ABC.incenter
z.O = T.ABC.circumcenter
T.BCI = triangle(z.B, z.C, z.I)
z.Ia = T.BCI.circumcenter
```



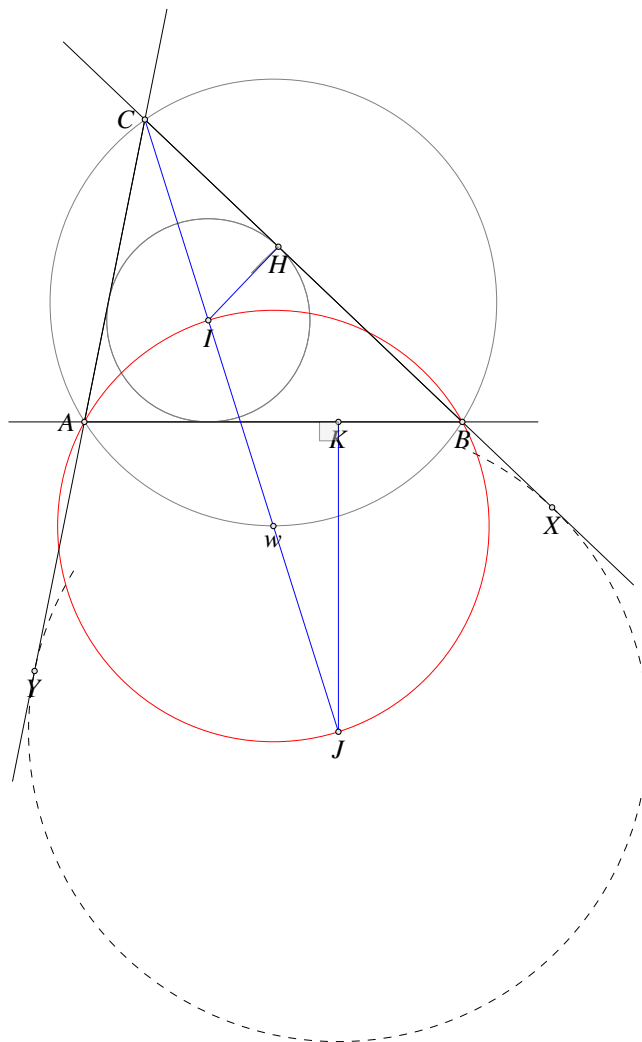
$\widehat{IaB} = \widehat{IaC}$ so (AIa) is a bisector of $\widehat{BAC}$
 Then $\widehat{ICB} = 1/2\widehat{IIaB} = 1/2\widehat{AIaB} = 1/2\widehat{ACB}$
 So (CI) is a bisector of $\widehat{ACB}$
 Conclusion: I is the incenter of ABC

11.13. Euler's relation

Let (Γ) be the circumscribed circle of center O to the triangle ABC . Let I and J be the centers of the inscribed and exscribed circles in the angle $\widehat{A}$ to the triangle ABC admitting r_1 and r_2 as respective radii

File euler.lua

```
init_elements()
z.A = point(0, 0)
z.B = point(5, 0)
z.C = point(0.8, 4)
T.ABC = triangle(z.A, z.B, z.C)
z.I = T.ABC.incenter
z.J, z.K = T.ABC.ex_circle(2):get()
_, z.H = T.ABC.in_circle():get()
z.X, z.Y, _ = T.ABC.projections(z.J)
z.O = T.ABC.circumcenter
T.IBA = triangle(z.I, z.B, z.A)
z.w = T.IBA.circumcenter
```



The quadrilateral $AIBJ$ having two opposite right angles is inscribed in a circle whose center ω is the middle of IJ . This center ω belongs to the perpendicular bisector of $[AB]$ and to the interior bisector of the angle $\widehat{ACB}$ and is thus the middle of the arc $\widehat{AB}$ south pole of the circle circumscribing the triangle ABC . (8.2).

Part 4. Inversion

12. Inversion

12.1. Definition

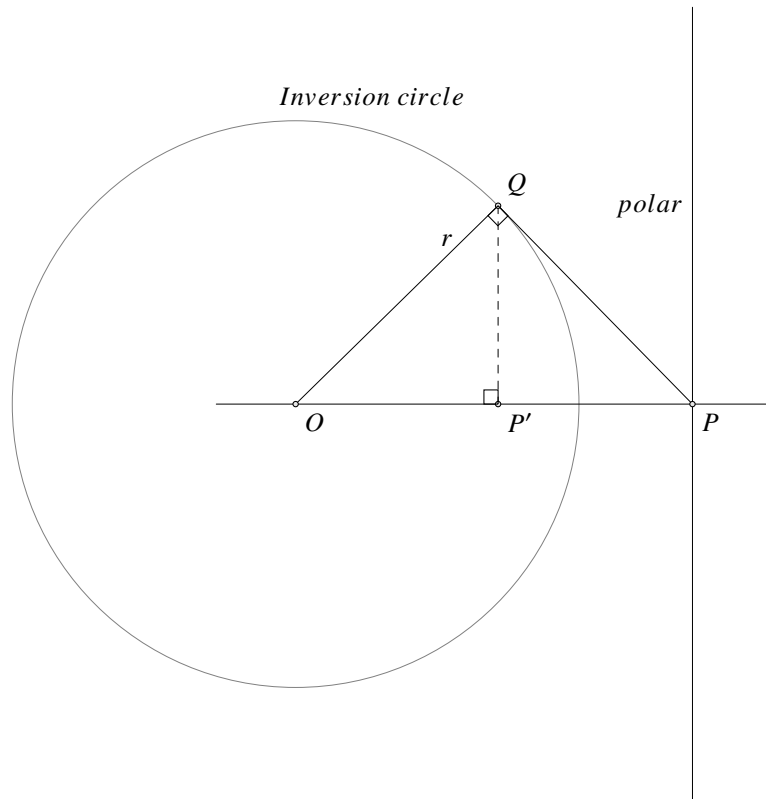
Definition 12.1. Inversion is the process of transforming points P to a corresponding set of points P' known as their inverse points. Two points P and P' are said to be inverses with respect to an inversion circle having inversion center O and inversion radius k if P' is the perpendicular foot of the altitude of $\triangle OPQ$, where Q is a point on the circle such that $OQ \perp PQ$.

If P and P' are inverse points, then the line L through P and perpendicular to OP is sometimes called a "polar" with respect to point P' , known as the "inversion pole". In addition, the curve to which a given curve is transformed under inversion is called its inverse curve (or more simply, its "inverse"). This sort of inversion was first systematically investigated by Jakob Steiner. [MathWorld]

Weissstein, Eric W. "Inversion." From MathWorld—A Wolfram Web Resource.

File def_inversion.lua:

```
init_elements()
z.O = point(0, 0)
z.P = point(7, 0)
z.T = point(3, 4)
C.inv = circle(z.O, z.T)
z.Q = C.inv:tangent_from(z.P).pb
z.Pp = C.inv:inversion(z.P)
L.OP = line(z.O, z.P)
L.polar = L.OP:orthogonal_from(z.P)
z.u, z.v = L.polar:get()
```



12.2. Inversion d'un cercle orthogonal

Theorem 12.2. *A circle orthogonal to the circle of inversion is globally invariant.*

Proof. Let M be a point on a circle (C) that is orthogonal to the circle of inversion (C_I) . The line (OM) intersects (C) at M' . Since (C) and C_I are orthogonal, we have $OM \cdot OM' = P(c)(O) = r^2$, where r denotes the radius of the circle of inversion. Therefore, $OM \cdot OM' = (\sqrt{k})^2 = k$. This means that M' , a point on (c) , is the image of M under the inversion T in question. The image of (c) under T is therefore contained within (c) , and by $a/$ we can conclude that this image is (c) itself. ■

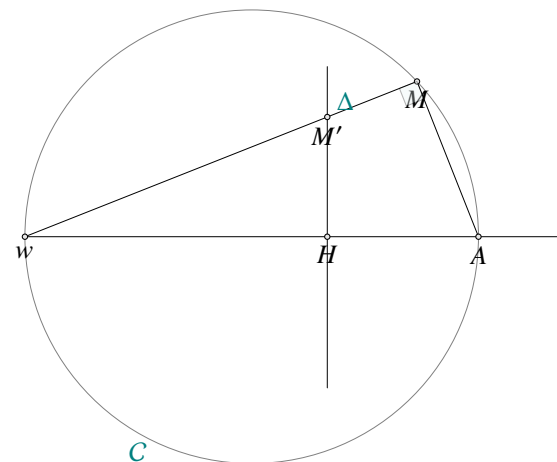
12.3. Cocyclic points

Theorem 12.3. *Cocyclic points Let i be an inversion of pole w and power k , and C a circle containing w . Consider the point A of C diametrically opposite w and its inverse A' by i . Let B be any point of C distinct from A and from w , of inverse B' . The equality $\overline{wA} \cdot \overline{wA'} = \overline{wB} \cdot \overline{wB'}$ (each member being equal to k) shows that the points A, A', B', B are cocyclic.*

Proof. To convince oneself of this, consider, for example, the circle γ circumscribed by the triangle AHM and the power of w with respect to this circle : $\overline{wA} \cdot \overline{wH} = \overline{wM} \cdot \overline{wN}$

where N is the other point common to the circle γ with the line (wM) . Then deduce that N is none other than M' . This being the case, since the angle $\widehat{AMM'}$ is right, so is the angle $\widehat{AHM'}$. Also, when M describes C , M' describes the line Δ perpendicular at H to the line (wA) . The image by i of the circle C (which contains the pole w) is the line Δ . The triangles wAB and $wA'B'$ are similar or the lines (AB) and $(A'B')$ are antiparallel. ■

File cocyclic.lua:



13. Midcircle

According to Eric Danneels and Floor van Lamoen:

A **midcircle** of two given circles is a circle that maps one to the other via an **inversion**. Midcircles belong to the same pencil of circles as the two originals. The center(s) of the midcircle(s) correspond to the *centers of similitude* of the given circles.

I have adopted the term **midcircle** used by Eric Danneels and Floor van Lamoen, as well as by Eric W. Weisstein, but many other names exist, such as **mid-circle** or **circle of antisimilitude** [Wikipedia]. The latter name can be found in *Advanced Euclidean Geometry* by Roger A. Johnson (2007). On the website *cut-the-knot.org*, reference is made to the **bisectal circle**, translated from French as **cercle bissecteur** and used by Hadamard, but this refers to the case where the two given circles intersect.

Four cases can be distinguished:

- (1) **The two circles intersect:** There are two midcircles, centered at the internal and external centers of similitude;
- (2) **One circle lies entirely inside the other:** There is a unique midcircle centered at the *internal* center of similitude;
- (3) **One circle lies entirely outside the other:** There is a unique midcircle centered at the *external* center of similitude;

- (4) **Tangency or single-point intersection:** These are limiting cases of the configurations above.

Let's examine at each of these cases in more detail:

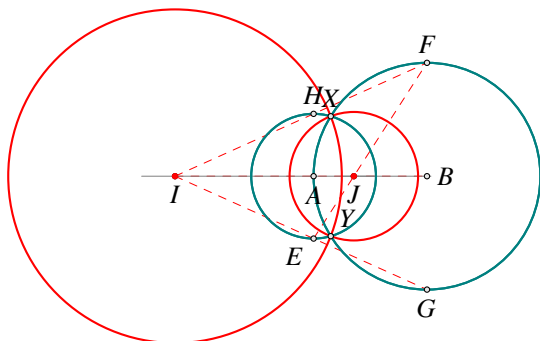
- (1) If the two given circles intersect, then there are two circles of inversion through their common points, with centers at the centers of similitudes. The two midcircles bisect their angles and are orthogonal to each other. The centers of the midcircles are the internal center of similitude and the external center of similitude I and J .

- Suppose $(\mathcal{A})$ and $(\mathcal{B})$ are two intersecting circles.
- Their common points define two inversion circles (midcircles), orthogonal to each other.
- The centers of these midcircles are the internal and external centers of similitude, denoted I and J .

To construct these centers:

- Take two diameters EH of $(\mathcal{A})$ and FG of $(\mathcal{B})$, such that the segments EH and FG are parallel.
- The intersection of lines (GE) and (AB) gives point J , the *external* center of similitude.
- The intersection of lines (EF) and (AB) gives point I , the *internal* center of similitude.

The circles (I) and (J) , centered respectively at I and J , are orthogonal and form the midcircles of $(\mathcal{A})$ and $(\mathcal{B})$. The division $(A, B; I, J)$ is harmonic.

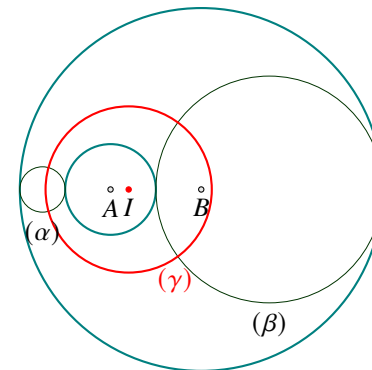


- (2) **One given circle lies entirely within the other.**

This configuration is slightly more delicate to handle. To construct the midcircle, we proceed as follows:

- First, construct two auxiliary circles (α) and (β) , each tangent to both of the given circles — one internally, one externally.
- Then, construct the *radical circle* orthogonal to both (α) and (β) .
- The center of this radical circle is the **radical center**, which coincides with the *internal center of similitude* of the original circles.

This radical circle is the unique midcircle in this case: it swaps the two original circles via inversion and lies in the same pencil of circles.



- (3) **The two given circles are external to each other.**

In this configuration, there exists a unique midcircle whose center is the **external center of similitude** of the two given circles.

Let I denote this external center of similitude. To construct the corresponding inversion circle (the midcircle), we proceed as follows:

- Construct the external center I based on the line joining the centers of the two given circles and their respective radii.
- Let E and F be the points of tangency (or auxiliary points) such that $IE \cdot IF = IH^2$.
- The point H lies on the desired midcircle, and the circle centered at I and passing through H is the unique **external midcircle**.

This circle performs an inversion that maps one circle onto the other and lies in the same pencil of circles.

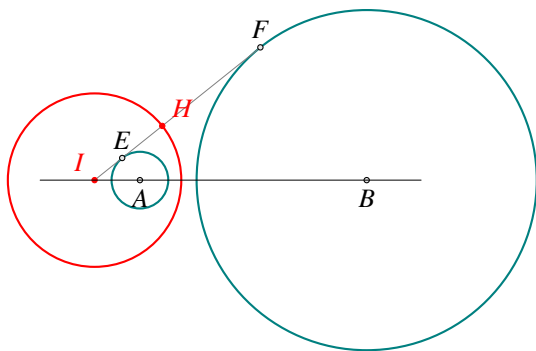
- (4) **The two given circles are external to each other.**

In this configuration, there exists a unique midcircle whose center is the **external center of similitude** of the two given circles.

Let I denote this external center of similitude. To construct the corresponding inversion circle (the midcircle), we proceed as follows:

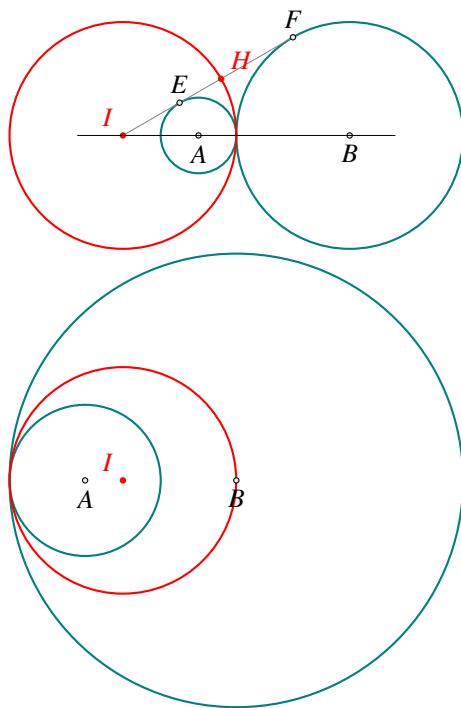
- Construct the external center I based on the line joining the centers of the two given circles and their respective radii.
- Let E and F be the points of tangency (or auxiliary points) such that $IE \cdot IF = IH^2$.
- The point H lies on the desired midcircle, and the circle centered at I and passing through H is the unique **external midcircle**.

This circle performs an inversion that maps one circle onto the other and lies in the same pencil of circles.



(5) The two circles are tangent.

- If circle (B) is *externally tangent* to circle (A) , the construction of the midcircle is identical to the case of two disjoint circles. The external center of similitude still exists, and the inversion circle centered at this point transforms one circle into the other.
- If one of the circles lies *inside* the other and they are *internally tangent*, the construction simplifies. The center of the midcircle is the internal center of similitude, which coincides with the point of tangency. The inversion circle is then centered at this point and passes through any auxiliary point satisfying the inversion relation.



Midcircle between a circle and a line

It is possible to generalize the notion of midcircle to the case of a circle and a line. A midcircle of a circle and a straight line is defined as a circle with respect to which the given circle and line are mutually inverse.

In this situation, there are only three possible cases, but they share several common features. The center of the midcircle lies on the line passing through the center of the given circle and perpendicular to the given line. Moreover, the center is also one of the intersection points of the circle and the line.

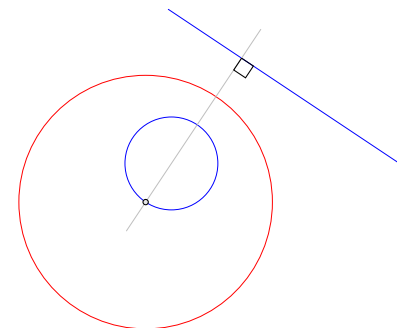
Three cases can be distinguished:

- The circle and the line intersect;
- The circle and the line are disjoint;
- The circle and the line are tangent.

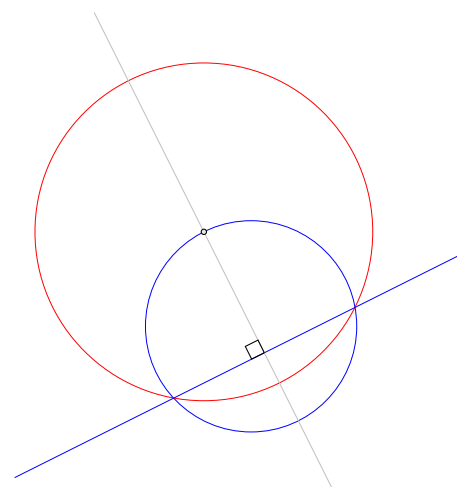
Let's look at each case:

Note: It should be noted that in each case, inversion allows us to verify that the two objects are inverses of each other.

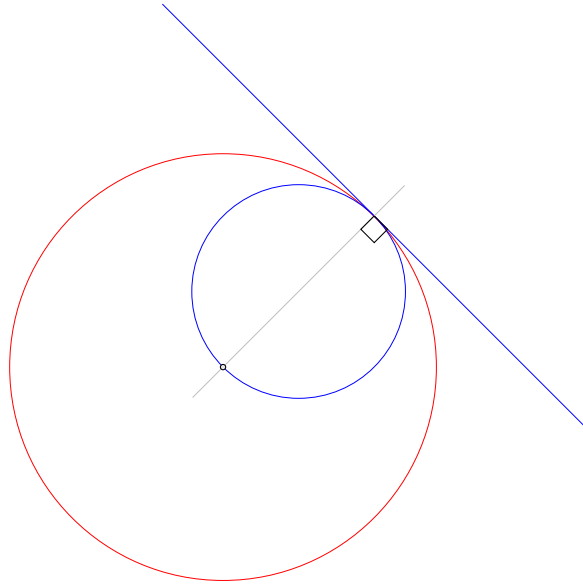
- The circle and the line are disjoint



- The circle and the line intersect

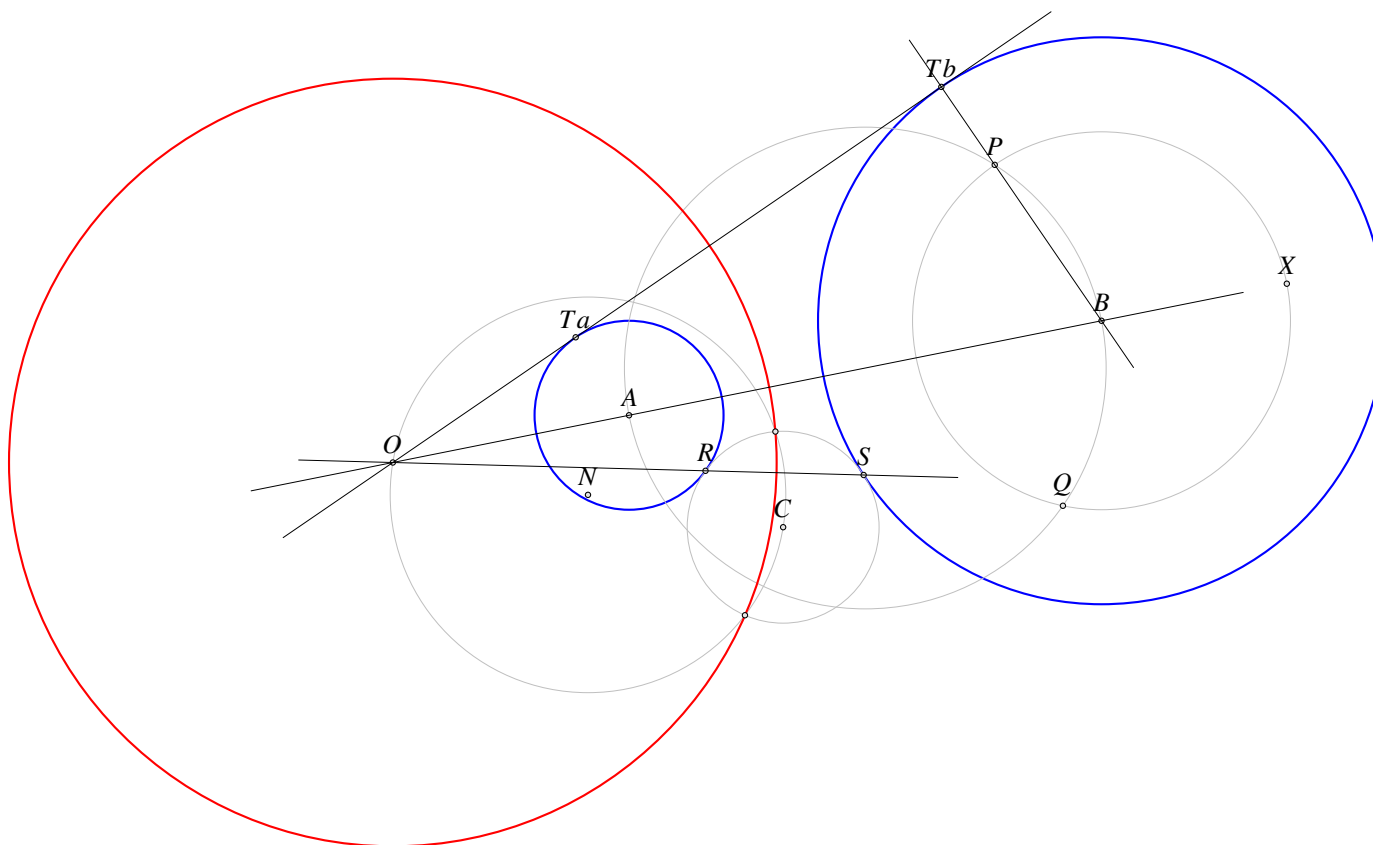


- The circle and the line are tangent



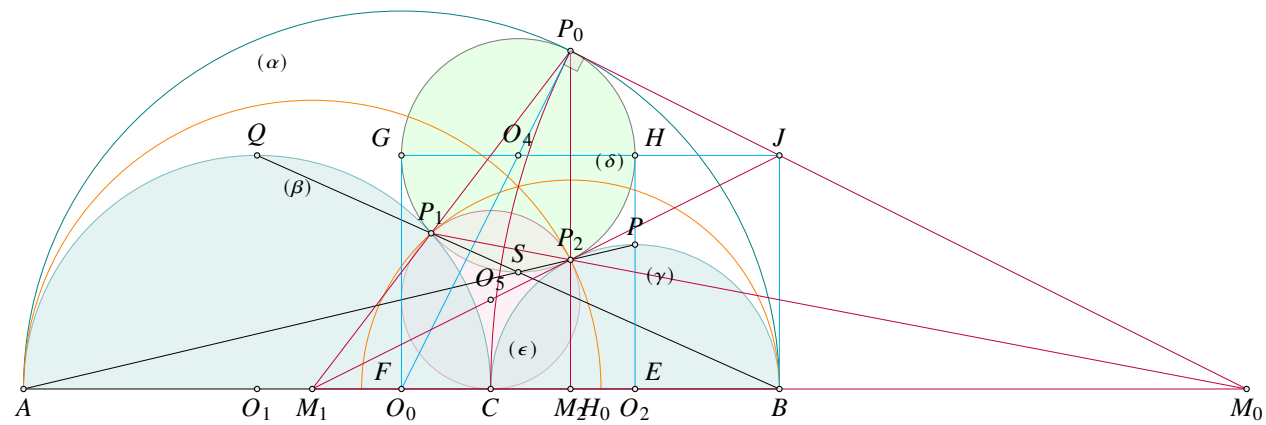
13.1. Construction of the midcircle of two disjoint circles

File `mid_circle.lua`:



File mid_circle_arbelos.lua:

File mid_circle_arbelos.lua:



Part 5. Special Circles

14. Apollonius Circles

14.1. Apollonius circles of a triangle

Definition 14.1. Apollonius circles in a triangle Let ABC be a triangle. For each side, say BC , and any positive real number $k \neq 1$, the Apollonius circle corresponding to the ratio k is the locus of points X in the plane such that

$$\frac{XB}{XC} = k.$$

When $k = \frac{AB}{AC}$, the circle is said to be the Apollonius circle of side BC in triangle ABC .

Each triangle has three such circles:

$$\mathcal{A}_A : \frac{XB}{XC} = \frac{AB}{AC}, \quad \mathcal{A}_B : \frac{XC}{XA} = \frac{BC}{BA}, \quad \mathcal{A}_C : \frac{XA}{XB} = \frac{CA}{CB}.$$

These are called the Apollonius circles of A , B , and C , respectively.

Main Properties.

- (1) Each Apollonius circle is orthogonal to the circumcircle of ABC .
- (2) The centers of the three Apollonius circles lie on the internal angle bisectors of the triangle.
- (3) The Apollonius circle of side BC passes through the vertex A if and only if the ratio is $k = \frac{AB}{AC}$, in which case it is the unique circle through A dividing the side BC internally in the same ratio:

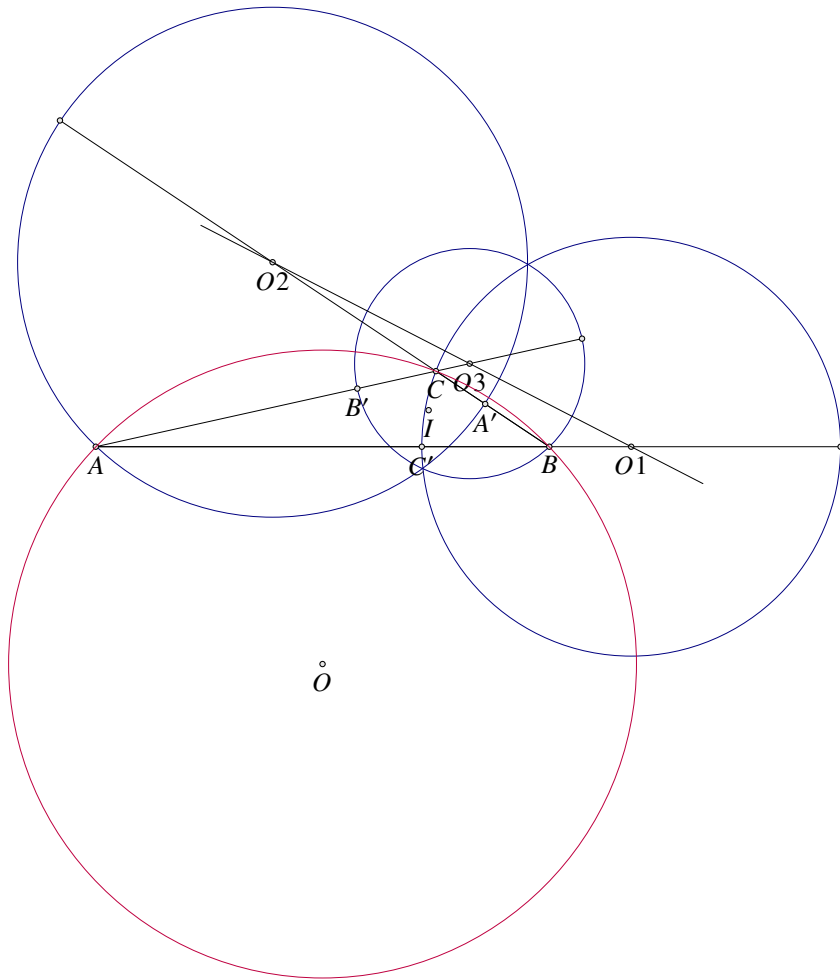
$$\frac{AB}{AC} = \frac{XB}{XC}.$$

- (4) The three Apollonius circles intersect pairwise in two points; each pair determines one of the isodynamic points of the triangle.
- (5) The radical axes of each Apollonius circle with the circumcircle are the internal angle bisectors of the triangle.
- (6) Under inversion in any Apollonius circle, the triangle is mapped to an equilateral triangle.

File `apollonius_triangle.lua`:

```
init_elements()
z.A = point(0, 0)
z.B = point(6, 0)
z.C = point(4.5, 1)
T.ABC = triangle(z.A, z.B, z.C)
z.I = T.ABC.incenter
z.O = T.ABC.circumcenter
L.CI = line(z.C, z.I)
```

```
z.Cp = intersection(T.ABC.ab, L.CI)
z.x = L.CI.north_pa
L.Cx = line(z.C, z.x)
z.R = intersection(L.Cx, T.ABC.ab)
L.CpR = line(z.Cp, z.R)
z.O1 = L.CpR.mid
L.AI = line(z.A, z.I)
z.Ap = intersection(T.ABC.bc, L.AI)
z.y = L.AI.north_pa
L.Ay = line(z.A, z.y)
z.S = intersection(L.Ay, T.ABC.bc)
L.ApS = line(z.Ap, z.S)
z.O2 = L.ApS.mid
L.BI = line(z.B, z.I)
z.Bp = intersection(T.ABC.ca, L.BI)
z.z = L.BI.north_pa
L.Bz = line(z.B, z.z)
z.T = intersection(L.Bz, T.ABC.ca)
L.Bpt = line(z.Bp, z.T)
z.O3 = L.Bpt.mid
```



Theorem 14.2 (Apollonius circles of a triangle: common points). *The three Apollonius circles of a (non-equilateral) triangle meet at exactly two points.*

Proof. Let ABC be a non-equilateral triangle, say with $AB \neq AC$. By definition, we have directly that if a point belongs to two of the circles, then it belongs to the third.

Indeed, if $\frac{MB}{MC} = \frac{AB}{AC}$ and $\frac{MA}{MB} = \frac{CA}{CB}$ then $\frac{MA}{MC} = \frac{BA}{BC}$

However, the circles are neither tangent nor disjoint if $AB \neq AC$. ■

Theorem 14.3 (Apollonius circles are coaxial). *The three Apollonius circles are coaxial. Their centers are aligned.*

$(O1C)$ is a tangent line to the circumscribed circle of triangle ABC . We deduce that the triangles $O1BC$ and $O1AC$ are similar which leads to

$\frac{O1B}{O1C} = \frac{BC}{AC} = \frac{a}{b} = \frac{O1C}{O1A}$ with $AB = c$, $BC = a$, $AC = b$. Thus $\frac{O1B^2}{O1C^2} = \frac{a^2}{b^2}$. But from power of the point $O1$ we have $O1C^2 = O1A \cdot O1B$ (This equality can also be obtained with relations in similar triangles).

This leads to $\frac{O1B}{O1A} = \frac{a^2}{b^2}$. Also $\frac{O1A}{O1C} = \frac{c^2}{a^2}$ and $\frac{O1C}{O1B} = \frac{b^2}{c^2}$.

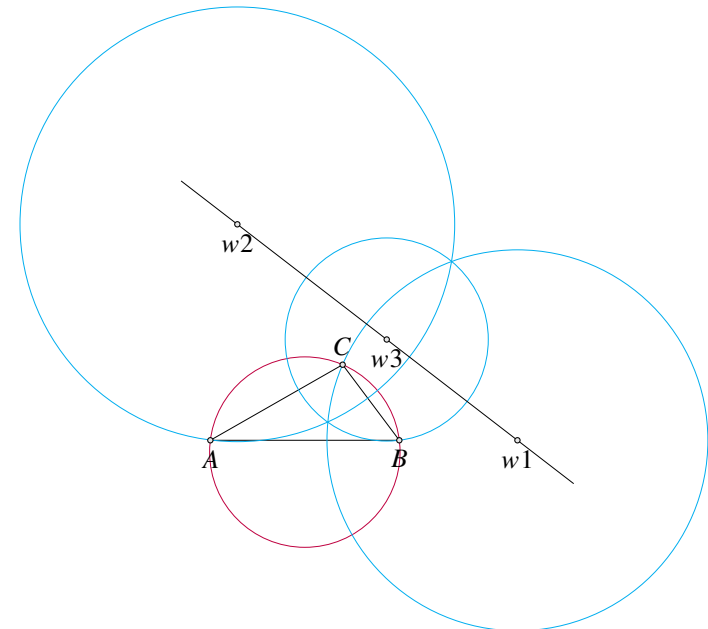
Multiplying the three expressions, from the converse of the Menelaus's theorem we conclude that $O1, O2$ and $O3$ are aligned and the three Apollonius circles are coaxial.

14.2. Apollonius circles in a triangle with method

Same result than the previous example using the function [T.ABC.ab : apollonius (k)]

File `apollonius_method.lua`:

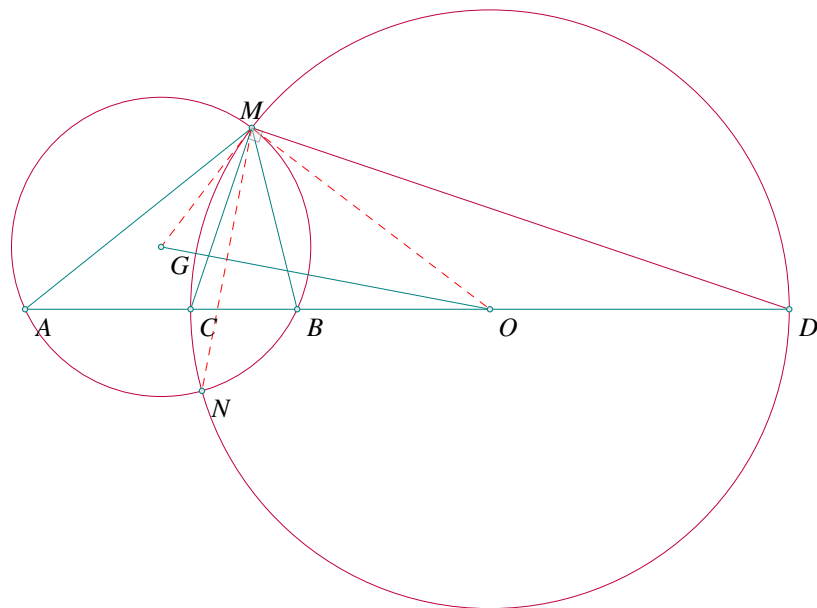
```
init_elements()
z.A = point(0, 0)
z.B = point(5, 0)
z.C = point(3.5, 2)
T.ABC = triangle(z.A, z.B, z.C)
z.O = T.ABC.circumcenter
C.AB = T.ABC.ab:apollonius(T.ABC.b / T.ABC.a)
z.w1, z.t1 = C.AB:get()
C.BC = T.ABC.bc:apollonius(T.ABC.c / T.ABC.b)
z.w2, z.t2 = C.BC:get()
C.AC = T.ABC.ca:apollonius(T.ABC.a / T.ABC.c)
z.w3, z.t3 = C.AC:get()
```



14.3. Apollonius circles and circumcircle of a triangle

Theorem 14.4 (Apollonius circles and circumcircle). *The apollonius circles of triangle are orthogonal to the circumcircle of this triangle.*

File apollonius_circum.lua:



Proof. $\widehat{OMB} + \widehat{BMC} = \widehat{OMC} = \widehat{OCM} = \widehat{CAM} + \widehat{AMC}$

or $\widehat{AMC} = \widehat{BMC}$ so $\widehat{OMB} = \widehat{CAM} = \widehat{BAM}$

Hence by the alternate segment theorem the result follows. (OM) is tangent at M to the circumcircle of the triangle ABC . ■

Proof. Another solution is to prove that $OM^2 = OB \cdot OA$ i.e. A and B are inverses with respect to the circle with diameter $[CD]$.

From $\frac{CA}{CB} = \frac{DA}{DB}$ and $OM = R$

$CB = R - OB$, $DB = R + OB$, $AD = R + OA$ and $AC = OA - R$

then $(OA - R)(OB + R) = (R - OB)(OA + R)$.

Finally after simplification

$R^2 = OM^2 = OA \cdot OB$

■

15. Soddy Circles

15.1. Soddy circles of a triangle

Theorem 15.1 (Soddy). *In geometry, the Soddy circles of a triangle are two circles associated with any triangle in the plane.[wikipedia]*

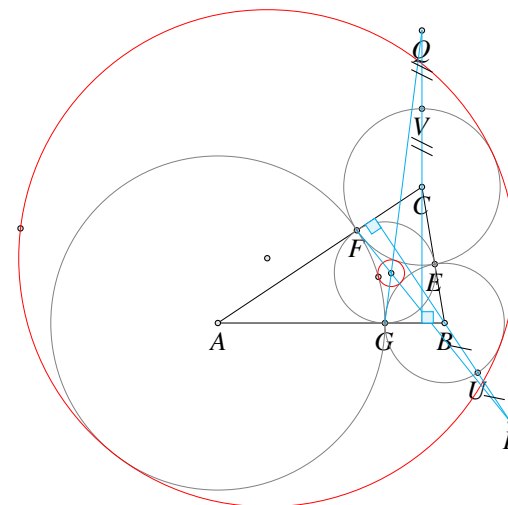
Given three noncollinear points, construct three tangent circles such that one is centered at each point and the circles are pairwise tangent to one another. Then there exist exactly two nonintersecting circles that are tangent to all three circles. These are called the inner and outer Soddy circles, and their centers are called the inner and outer Soddy centers, respectively.

Weisstein, Eric W. "Soddy Circles." From MathWorld—A Wolfram Web Resource

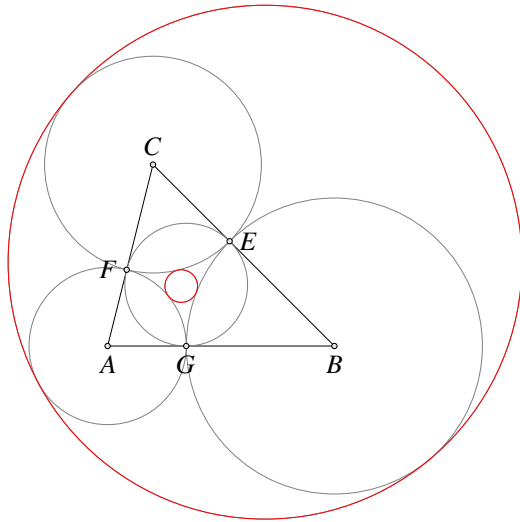
15.2. Soddy circle without function

The following construction is inspired by that of the following site:

[Mathafou](#)



15.3. Soddy circle with function



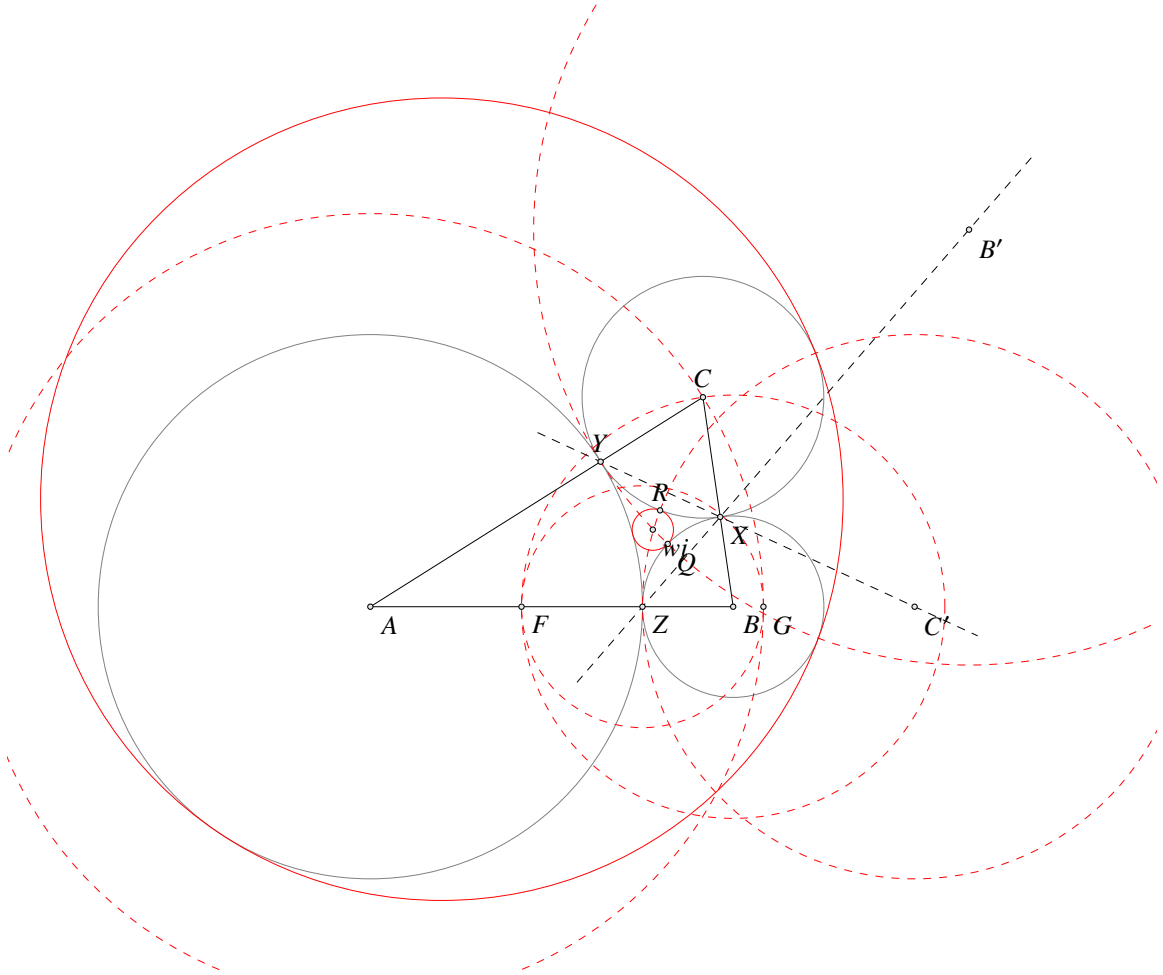
15.4. Construction of Soddy's circles

Proposition 1. Given two mutually tangent circles with centers A and B , and the line (AB) , a generic circle tangent to both circles is constructible in five moves. Articles:

Efficiently Constructing Tangent Circles Arthur Baragar & Alex Kontorovich

Mathematics Magazine

File soddy.lua:



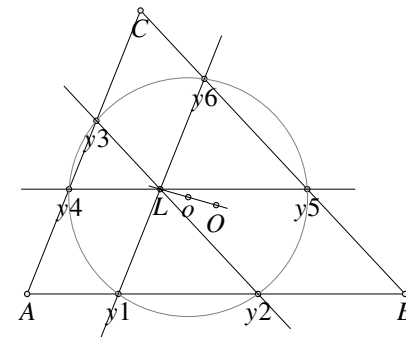
16. Various circles

16.1. First Lemoine circle

Draw lines through the symmedian point L and parallel to the sides of the triangle. The points where the parallel lines intersect the sides of the triangle then lie on a circle known as the first Lemoine circle. It has center at the Brocard midpoint, i.e., the midpoint of $[OL]$, where O is the circumcenter and K is the symmedian point

[Weisstein, Eric W. "First Lemoine Circle." From MathWorld—A Wolfram Web Resource.](#)

File `first_lemoine.lua`:

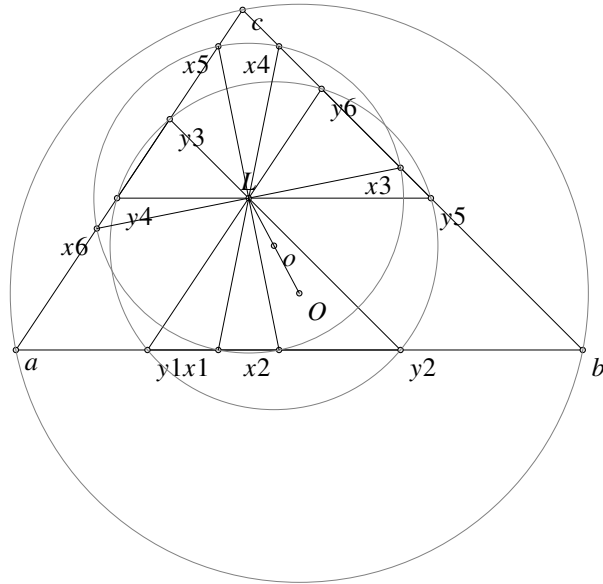


16.2. First and second Lemoine circles

Draw antiparallels through the symmedian point L . The points where these lines intersect the sides then lie on a circle, known as the cosine circle (or sometimes the second Lemoine circle).

[Weisstein, Eric W. "Cosine Circle." From MathWorld—A Wolfram Web Resource.](#)

File `second_lemoine.lua`:

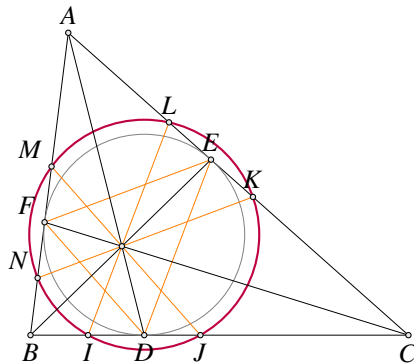


16.3. Adams's circle

Given a triangle ABC , construct the contact triangle. Now extend lines parallel to the sides of the contact triangle from the Gergonne point. These intersect the triangle ABC in the six points M, N, I, J, K, L . Adams proved in 1843 that these points are concyclic in a circle now known as the Adams' circle.

Weisstein, Eric W. "Adams' Circle." From MathWorld—A Wolfram Web Resource.

File adam.lua:

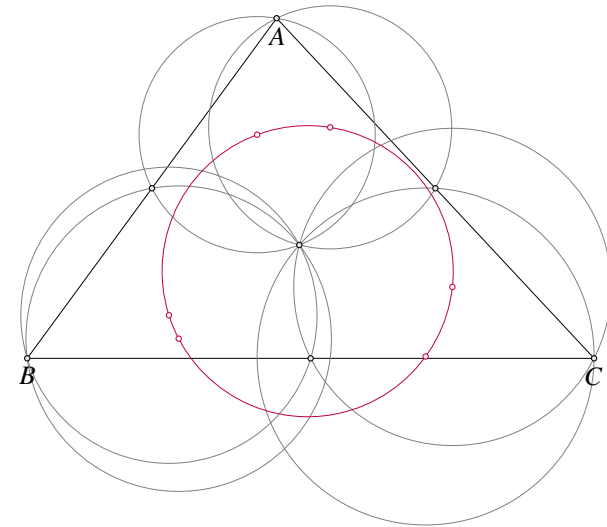


16.4. Van Lamoen's circle

Divide a triangle by its three medians into six smaller triangles. Surprisingly, the circumcenters of the six circumcircles of these smaller triangles are concyclic. Their circumcircle is known as the van Lamoen circle.

Weisstein, Eric W. "van Lamoen Circle." From MathWorld—A Wolfram Web Resource.

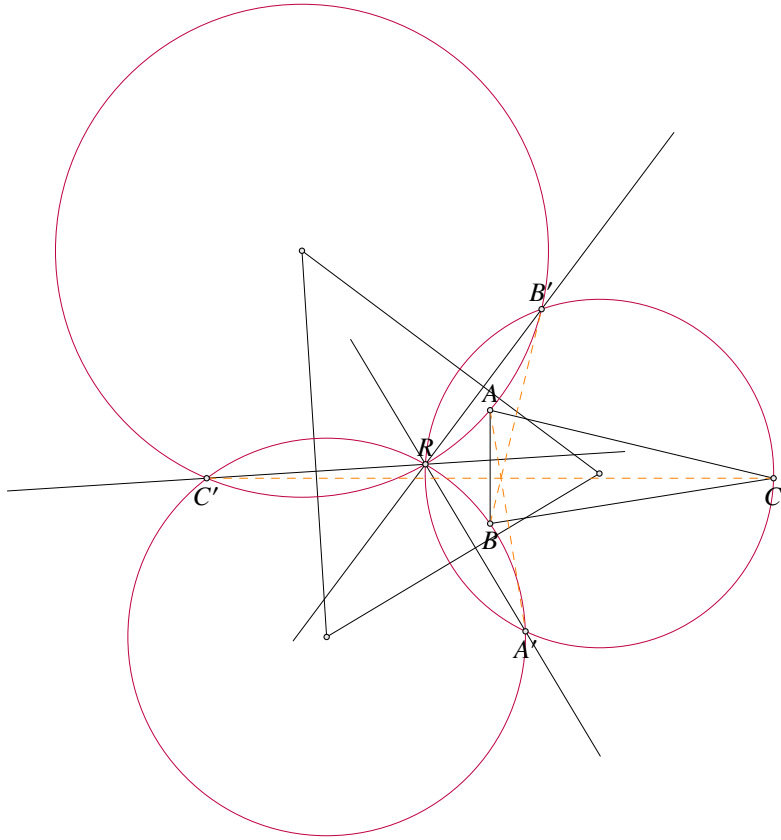
File lamoen.lua:



16.5. Yiu's circles variant one

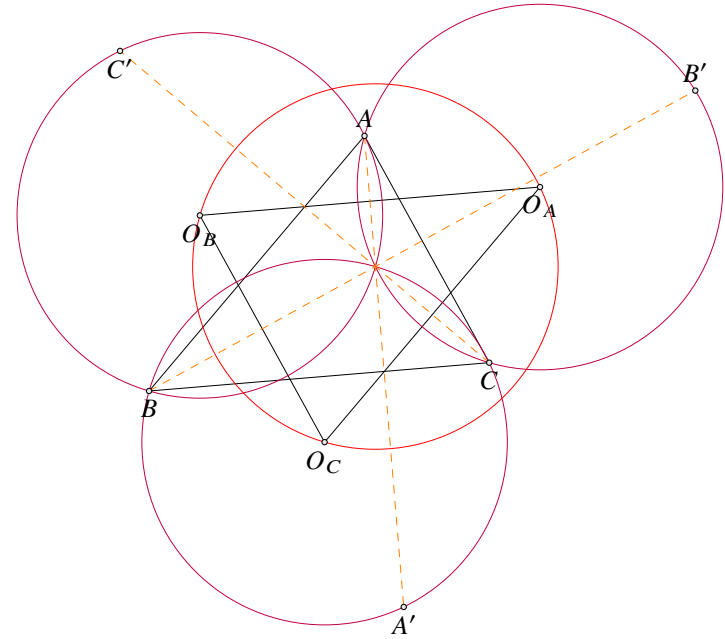
The Yiu circles of a reference triangle is the circles passing through one vertex and the reflections of other vertices with respect to the opposite sides. All three circles have one thing in common: their radical center.

File yiu1.lua:



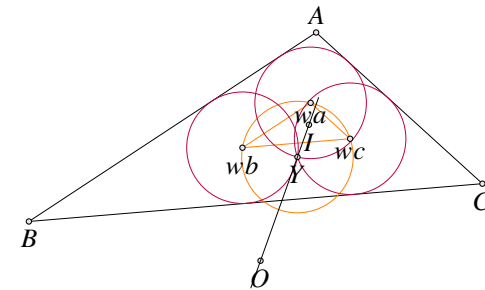
16.6. Yiu's circles variant two

File yiu2.lua:



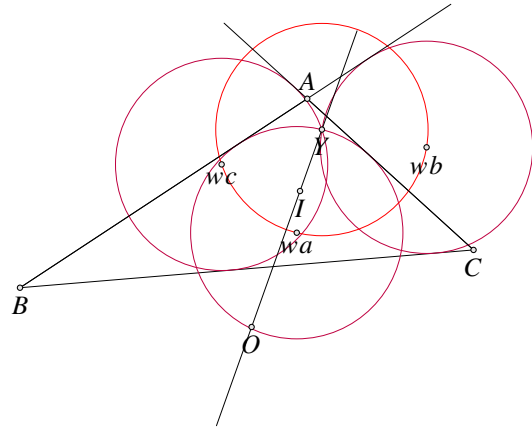
16.7. Yff's circles variant one

File yff1.lua:



16.8. Yff's circles variant two

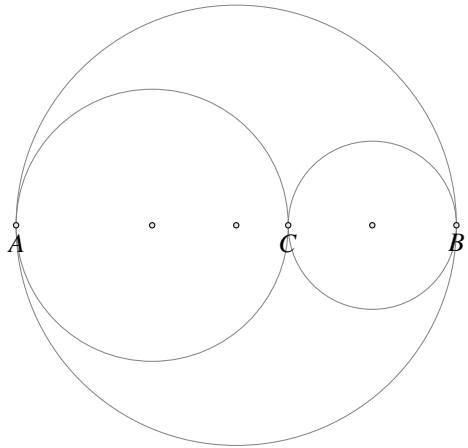
File yff2.lua:



17. Arbelos and Related Figures

17.1. Gold Arbelos

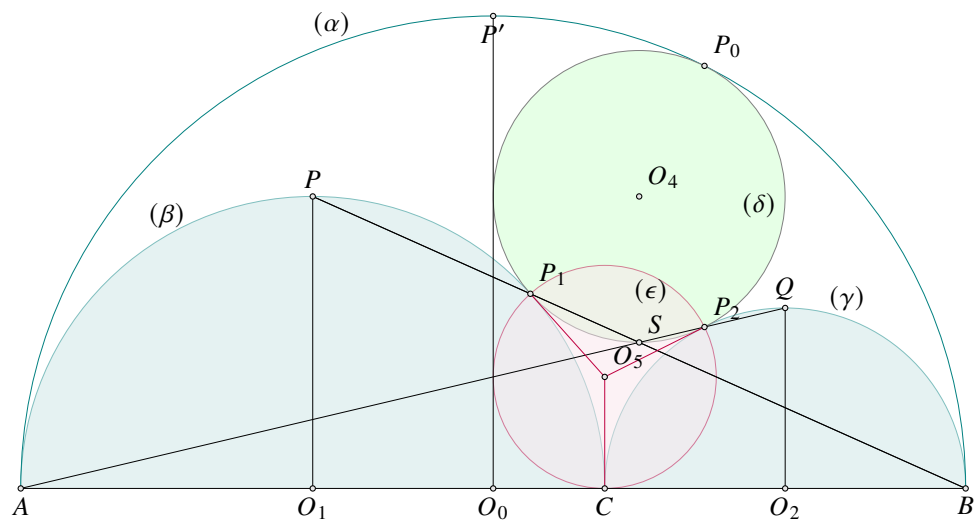
File `gold_arbelos.lua`:



17.2. Bankoff circle

Definition 17.1. Bankoff circle In geometry, the Bankoff circle or Bankoff triplet circle is a certain Archimedean circle that can be constructed from an arbelos; an Archimedean circle is any circle with area equal to each of Archimedes' twin circles. The Bankoff circle was first constructed by Leon Bankoff in 1974.[Wikipedia]

File `bankoff.lua`:

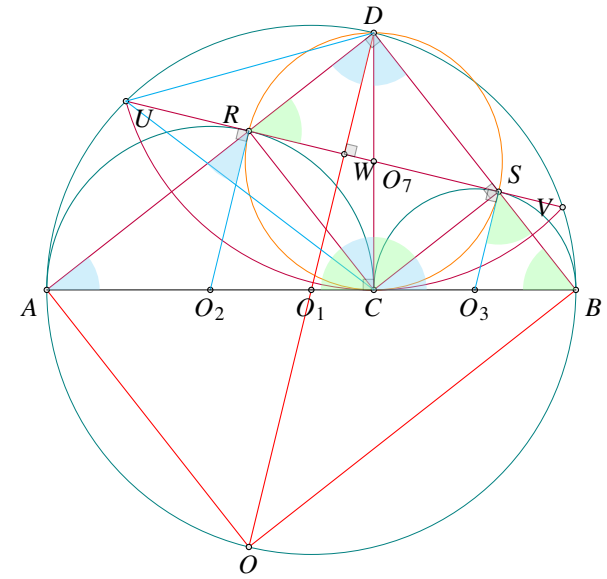
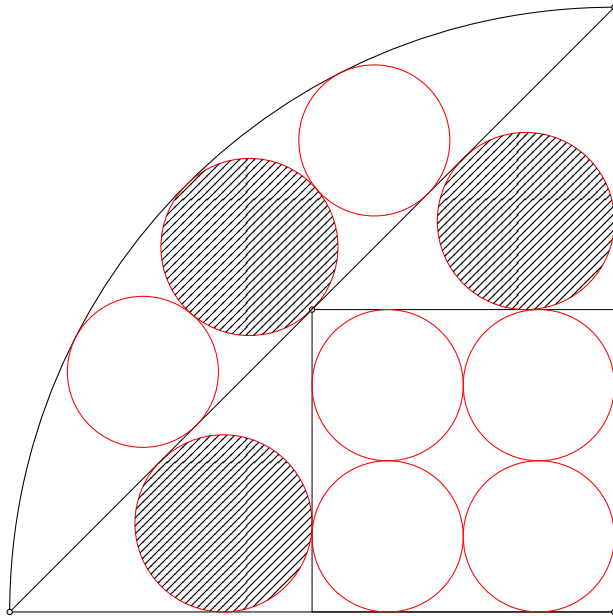


17.3. Related circles from Leon Bankoff

The shaded circles are equal and the unshaded circles are equal.

File `bankoff.lua`:

Let ABC be an acute-angled triangle, neither right-angled nor equilateral.



17.4. Gold Arbelos properties

File `gap.lua`:


```

-- cercles tangent à Lba et Lbc et passant par C.center
local pc, pa = Lba:LLP(Lbc, C.center)

-- petite fonction utilitaire : construit le candidat n
local function candidate(idcx)
  local w = pc:get(idcx)
  if not w then return nil end
  local t = T.bc:projection(w)
  local r_new = w:length(t)          -- rayon du nouveau cercle
  local d      = w:length(C.center)  -- distance entre centres
  return w, t, r_new, d
end

local w1, t1, r1, d1 = candidate(1)
local w2, t2, r2, d2 = candidate(2)

local EPS = tkz.epsilon or 1e-6

-- test de tangence externe avec C
local function good_ext(d, r_new)
  return math.abs(d - (r_new + C.radius)) < 10 * EPS
end

-- choix du bon candidat
if w1 and good_ext(d1, r1) and not (w2 and good_ext(d2, r2)) then
  return w1, t1
elseif w2 and good_ext(d2, r2) and not (w1 and good_ext(d1, r1)) then
  return w2, t2
elseif w1 and w2 then
  -- si les deux ou aucun ne passent le test, on prend par défaut
  -- le cercle de plus petit rayon (habituellement le cercle intérieur)
  if r1 <= r2 then
    return w1, t1
  else
    return w2, t2
  end
else
  -- pas de solution exploitable
  return nil, nil
end
end

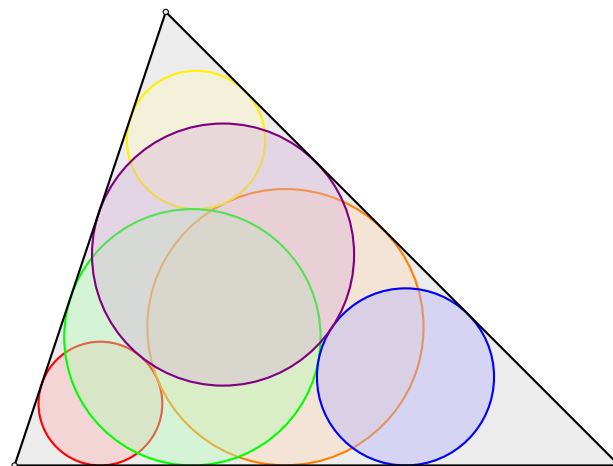
L.ba = T.ABC:bisector()
z.c1 = L.ba:report(r)
z.t1 = T.ABC.ab:projection(z.c1)
C.last = circle:new(z.c1, z.t1)

```

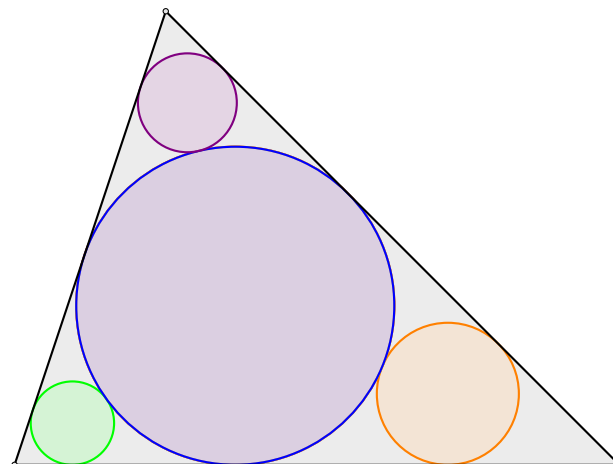
```

local vertices = { "A", "B", "C" }
for i = 2, 6 do
  T.used = triangle:new(
    z[vertices[math.fmod(i - 2, 3) + 1]],
    z[vertices[math.fmod(i - 1, 3) + 1]],
    z[vertices[math.fmod(i, 3) + 1]]
  )
  z["c" .. i], z["t" .. i] = newcircle(T.used, C.last)
  C.last = circle:new(z["c" .. i], z["t" .. i])
end

```



When the first circle is the incircle then you get only 4 circles.

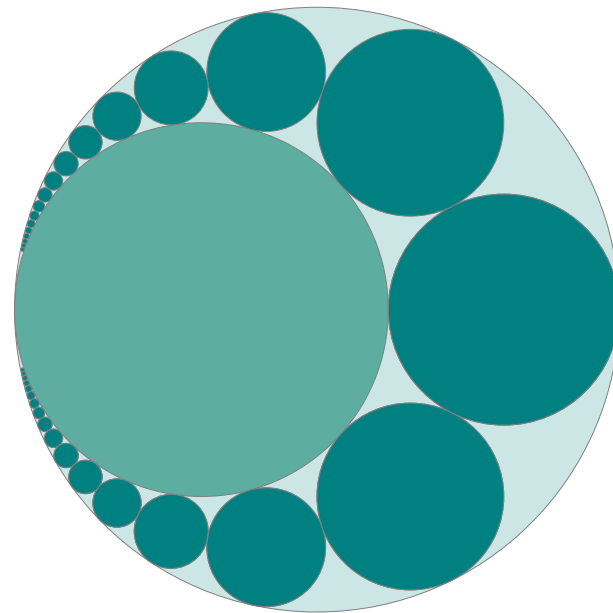


18.2. Pappus chain

Definition 18.2 (Pappus chain). Starting with the circle P_1 tangent to the three semicircles forming the arbelos, construct a chain of tangent circles P_i , all tangent to one of the two small interior circles and to the large exterior one. This chain is called the Pappus chain

File pappus.lua:

```
init_elements()
xC, nc = 10, 16
xB = xC * tkz.invpphi
xD = (xC * xC) / xB
xJ = (xC + xD) / 2
r = xD - xJ
z.A = point(0, 0)
z.B = point(xB, 0)
z.C = point(xC, 0)
L.AC = line(z.A, z.C)
z.i = L.AC.mid
L.AB = line(z.A, z.B)
z.j = L.AB.mid
z.D = point(xD, 0)
C.AC = circle(z.A, z.C)
for i = -nc, nc do
  z["J" .. i] = point(xJ, 2 * r * i)
  z["H" .. i] = point(xJ, 2 * r * i - r)
  z["J" .. i .. "p"],
  z["H" .. i .. "p"] = C.AC:inversion(z["J" .. i], z["H" .. i])
  L.AJ = line(z.A, z["J" .. i])
  C.JH = circle(z["J" .. i], z["H" .. i])
  z["S" .. i], z["T" .. i] = intersection(L.AJ, C.JH)
  z["S" .. i .. "p"],
  z["T" .. i .. "p"] = C.AC:inversion(z["S" .. i], z["T" .. i])
  L.SpTp = line(z["S" .. i .. "p"], z["T" .. i .. "p"])
  z["I" .. i] = L.SpTp.mid
end
```

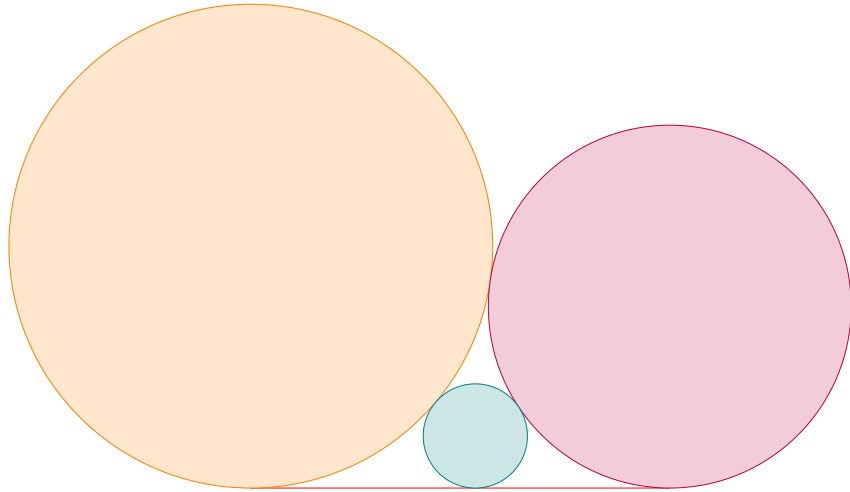


18.3. Three Circles

Construct a circle tangent to two circles tangent to each other and to a straight line.

File three_circles.lua:

```
init_elements()
function threecircles(c1, r1, c2, r2, c3, h1, h3, h2)
  local xk = math.sqrt(r1 * r2)
  local cx = (2 * r1 * math.sqrt(r2)) / (math.sqrt(r1) + math.sqrt(r2))
  local cy = (r1 * r2) / (math.sqrt(r1) + math.sqrt(r2)) ^ 2
  z[c2] = point(2 * xk, r2)
  z[h2] = point(2 * xk, 0)
  z[c1] = point(0, r1)
  z[h1] = point(0, 0)
  L.h1h2 = line(z[h1], z[h2])
  z[c3] = point(cx, cy)
  z[h3] = L.h1h2:projection(z[c3])
end
threecircles("A", 4, "B", 3, "C", "E", "G", "F")
```



Part 6. Projective Geometry

19. Harmonic Division

19.1. Definitions

Definition 19.1. [Cross Ratio] Let A, B, C , and D be four collinear points. The *cross ratio* is defined by

$$(A, B; C, D) = \frac{\overline{AC}}{\overline{BC}} \div \frac{\overline{AD}}{\overline{BD}} = \frac{\overline{AC} \cdot \overline{BD}}{\overline{AD} \cdot \overline{BC}}.$$

Definition 19.2. [Harmonic Division] Four collinear points A, B, C , and D form a *harmonic division* if

$$(A, B; C, D) = -1.$$

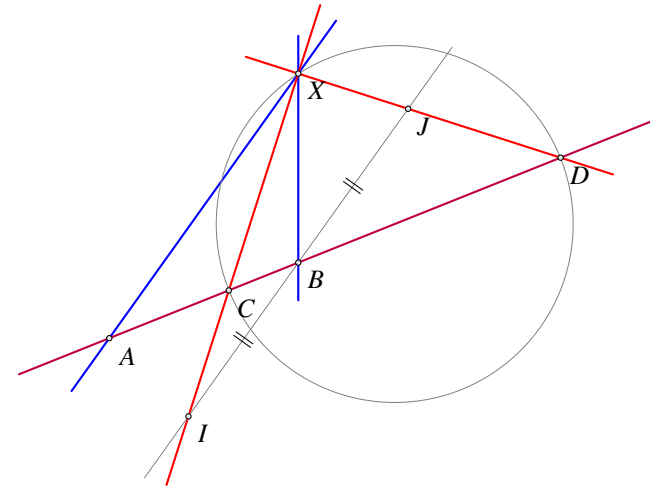
In this case, the points C and D are said to be *harmonic conjugates* with respect to A and B .

Definition 19.3. Let A, B, C , and D be four distinct collinear points on a line d , and let X be a point not lying on d . The pencil of lines

$$(XA), (XB), (XC), (XD)$$

is said to be *harmonic* if

$$(A, B; C, D) = -1.$$



Proposition 19.4. A pencil of four lines is harmonic if and only if a line parallel to one of its rays is divided by the other three into two equal segments.

Proof. We use directed lengths. Let the line through B parallel to XA meet XC at I and XD at J . Since $BI \parallel XA$, Thales' theorem applied in triangle XCA gives

$$\frac{\overline{BI}}{\overline{XA}} = \frac{\overline{BC}}{\overline{AC}}.$$

Similarly, since $BJ \parallel XA$, Thales' theorem applied in triangle XDA gives

$$\frac{\overline{BJ}}{\overline{XA}} = \frac{\overline{BD}}{\overline{AD}}.$$

Dividing these two equalities, we obtain

$$\frac{\overline{BI}}{\overline{BJ}} = \frac{\overline{BC}}{\overline{AC}} \cdot \frac{\overline{AD}}{\overline{BD}}.$$

But

$$(A, B; C, D) = \frac{\overline{AC}}{\overline{BC}} \cdot \frac{\overline{BD}}{\overline{AD}}.$$

Hence

$$\frac{\overline{BI}}{\overline{BJ}} = \frac{1}{(A, B; C, D)}.$$

Therefore,

$$(A, B; C, D) = -1 \iff \frac{\overline{BI}}{\overline{BJ}} = -1.$$

This last equality means precisely that I and J lie on opposite sides of B and that

$$BI = BJ.$$

Thus B is the midpoint of $[IJ]$.

Consequently,

$$(A, B; C, D) = -1 \iff B \text{ is the midpoint of } [IJ].$$

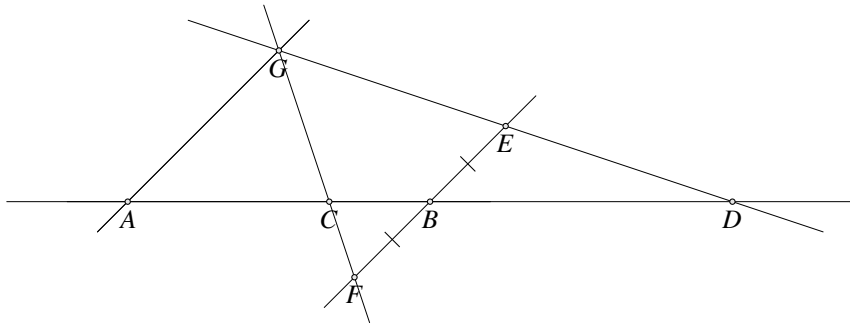
19.2. Construction of a harmonic division

Given three distinct collinear points A , C , and B , with C not the midpoint of AB , the harmonic conjugate D of C with respect to A and B may be constructed as follows. We'll use the previous result:

Choose a point G not on line AB . The line through B parallel to GA meets GC at a point F . Let E be the reflection of F across B ; then the line GF intersects AB at a point D , which is the desired harmonic conjugate. Indeed, the configuration produces a harmonic bundle $(G, A; B, F)$, whose image under the symmetry with center B yields $(G, A; B, E)$; projecting from G gives the harmonic quadruple $(A, B; C, D)$.

If GE were parallel to AB , the quadrilateral $AGBE$ would be a parallelogram, forcing C to be the midpoint of AB , contrary to assumption. Thus the construction ensures both existence and uniqueness. When C is the midpoint of AB , the line GE becomes parallel to AB , and the harmonic conjugate reduces to the point at infinity on line AB , in agreement with projective geometry.

File harmonic_div1.lua:



19.3. Harmonic conjugate of a point outside a segment

Theorem 19.5. How to get the harmonic conjugate C of a point D outside the segment $[AB]$.

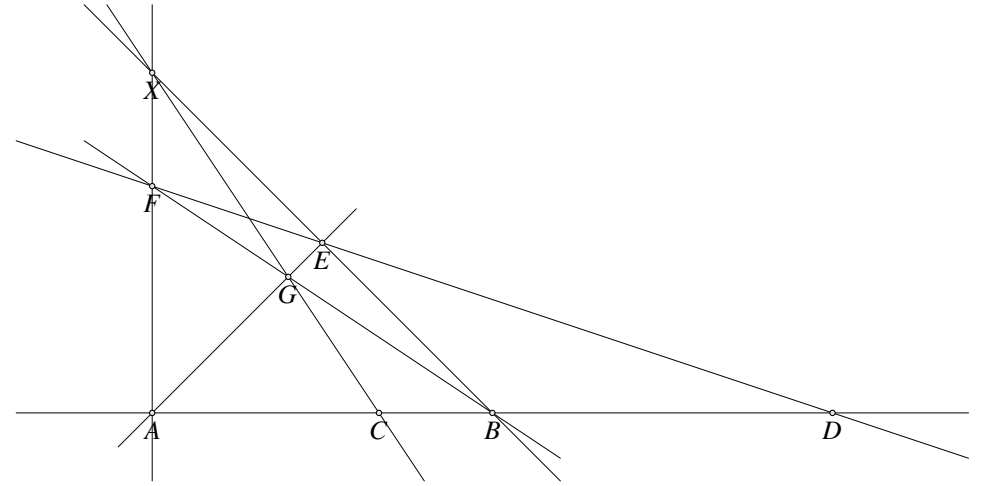
Proof. We choose a point X not aligned with A and B . Here I choose X such that (AX) is orthogonal to (AB) and with $AB = AX$. Then I denote by E the middle of $[XB]$.

F is the intersection of (DE) with (XA) , then G is the intersection of (AE) with (BF) and finally C is the intersection of lines (XG) and (AB) ■

File harmonic.lua:

```
init_elements()
z.A = point(0, 0)
z.B = point(6, 0)
z.D = point(12, 0)
L.AB = line(z.A, z.B)
z.X = L.AB.north_pa
L.XB = line(z.X, z.B)
z.E = L.XB.mid
L.ED = line(z.E, z.D)
L.AX = line(z.A, z.X)
```

```
L.AE = line(z.A, z.E)
z.F = intersection(L.ED, L.AX)
L.BF = line(z.B, z.F)
z.G = intersection(L.AE, L.BF)
L.GX = line(z.G, z.X)
z.C = intersection(L.GX, L.AB)
```



19.4. Harmonic division with golden ratio

Example 19.6 (Harmonic division with golden ratio). Two line segments of lengths a and b are said to be in the *golden ratio* if

$$\frac{a+b}{a} = \frac{a}{b} = \varphi.$$

Its principal representations are:

$$\varphi \approx 1.618033988749894 \dots$$

$$\varphi = \frac{1 + \sqrt{5}}{2}$$

$$\varphi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}$$

The golden ratio was known to Euclid as the *extreme and mean ratio*, and was later referred to as the *divine proportion* by Luca Pacioli.

File: gold.lua:



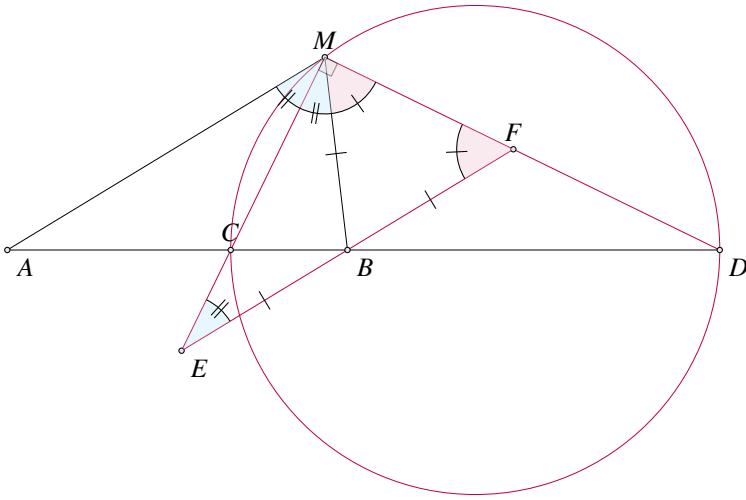
19.5. Bisector and harmonic division

Theorem 19.7 (Bisector and harmonic division). *Let four points A, C, B and D , in this order, lying on the straight line (d) and M un point pris hors de (d) . Then, if two of the following three propositions are true, then the third is also true:*

- (1) *The division $(A,B;C,D)$ is harmonic. $(CA/CB = DA/DB)$*
- (2) *(MC) is the internal angle bisector of $\widehat{AMB}$.*
- (3) *$(MD) \perp (MC)$.*

File bisector.lua:

```
init_elements()
z.A = point(0, 0)
z.B = point(6, 0)
z.M = point(5.6, 3.4)
T.AMB = triangle(z.A, z.M, z.B)
L.AB = T.AMB.ca
L.bis = T.AMB.bisector(1)
z.C = L.bis.pb
L.bisext = T.AMB.bisector_ext(1)
z.D = intersection(L.bisext, L.AB)
L.CD = line(z.C, z.D)
z.O = L.CD.mid
L.AM = line(z.A, z.M)
L.LL = L.AM:ll_from(z.B)
L.MC = line(z.M, z.C)
L.MD = line(z.M, z.D)
z.E = intersection(L.LL, L.MC)
z.F = intersection(L.LL, L.MD)
```



Proof. (2) and (3) involve (1)

C point of intersection of the internal bisector of $\widehat{AMB}$, D point of intersection of the external bisector of $\widehat{AMB}$.

We draw the parallel to (AM) passing through B . It cuts (CM) at E and (MD) at F .

On one hand $\frac{CA}{CB} = \frac{AM}{BE} = \frac{MA}{MB}$ on the other hand $\frac{DA}{DB} = \frac{AM}{BF} = \frac{MA}{MB}$

It follows: $\frac{CA}{CB} = \frac{DA}{DB} = \frac{MA}{MB}$

We conclude that $(A, B; C, D)$ is harmonic.

and the lines (MA) , (MB) , (MC) and (MD) form a harmonic pencil.

The circle of diameter $[CD]$ is the circle of Apollonius for the triplet (A, B, k) with $k = \frac{CA}{CB}$. ■

20. Pascal's Theorem

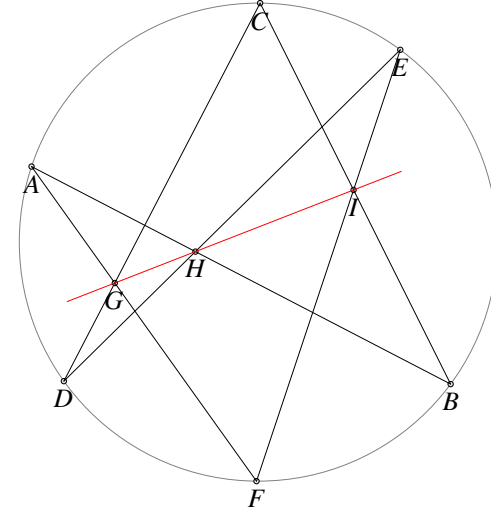
20.1. Definitions

Definition 20.1. A *conic* is a curve obtained as the intersection of a plane with a cone.

Definition 20.2. A hexagon is said to be *inscribed in a conic* if all its vertices lie on the conic.

20.2. Main Result

Theorem 20.3 (Pascal's Theorem (version circle)). *Given six points (which may coincide) on the circumference of a circle, labelled A, C, E, B, F , and D in that order around the circle, the intersection points of AB and DE , AF and CD , and BC and EF are collinear.*



Proof. Let G be the intersection of $\overline{CD}$ and $\overline{FA}$, let H be the intersection of $\overline{AB}$ and $\overline{DE}$, and let I be the intersection of $\overline{BC}$ and $\overline{EF}$. We will prove that these three points are collinear. Let U be the intersection of $\overline{CD}$ and $\overline{EF}$, let V be the intersection of $\overline{AB}$ and $\overline{EF}$, and let W be the intersection of $\overline{AB}$ and $\overline{CD}$.

By Menelaus in $\triangle UVW$ with line HDE , we have

$$\frac{VH}{WH} \cdot \frac{WD}{UD} \cdot \frac{UE}{VE} = 1.$$

By Menelaus in $\triangle UVW$ with line AGF , we have

$$\frac{VA}{WA} \cdot \frac{WG}{UG} \cdot \frac{UF}{VF} = 1.$$

By Menelaus in $\triangle UVW$ with line BCI , we have

$$\frac{VB}{WB} \cdot \frac{WC}{UC} \cdot \frac{UI}{VI} = 1.$$

Multiplying these equalities, we obtain

$$\frac{VH}{WH} \cdot \frac{WD}{UD} \cdot \frac{UE}{VE} \cdot \frac{VA}{WA} \cdot \frac{WG}{UG} \cdot \frac{UF}{VF} \cdot \frac{VB}{WB} \cdot \frac{WC}{UC} \cdot \frac{UI}{VI} = 1.$$

Rearranging, we get

$$\frac{WD}{UD} \cdot \frac{UE}{VE} \cdot \frac{VA}{WA} \cdot \frac{UF}{VF} \cdot \frac{VB}{WB} \cdot \frac{WC}{UC} \cdot \frac{VH}{WH} \cdot \frac{WG}{UG} \cdot \frac{UI}{VI} = 1.$$

By the power of a point, we have

$$\frac{WD}{UD} \cdot \frac{UE}{VE} \cdot \frac{VA}{WA} \cdot \frac{UF}{VF} \cdot \frac{VB}{WB} \cdot \frac{WC}{UC} = \frac{WD \cdot WC}{WA \cdot WB} \cdot \frac{VA \cdot VB}{VE \cdot VF} \cdot \frac{UE \cdot UF}{UC \cdot UD} = 1.$$

Thus, the previous product simplifies to

$$\frac{VH}{WH} \cdot \frac{WG}{UG} \cdot \frac{UI}{VI} = 1.$$

By Menelaus, the points G , H , and I are collinear. ■

Theorem 20.4 (Pascal's Theorem - general). *Let A , B , C , D , E , and F be six points on a conic. Then the intersection points*

$$AB \cap DE, \quad BC \cap EF, \quad CD \cap FA$$

are collinear.

Projective reduction to the circle. We reformulate Pascal's reasoning in modern terms.

- (1) Start with six points on a circle and consider the hexagon formed by these points.
- (2) Apply a central projection from a point onto another plane. Under this projection, the circle is mapped to a general conic.
- (3) The six points are mapped to six points on this conic, hence the hexagon becomes a hexagon inscribed in the conic.
- (4) Incidence properties are preserved under projection: in particular, intersections of lines are mapped to intersections, and collinearity is preserved.

Therefore, if Pascal's theorem holds for a hexagon inscribed in a circle, it also holds for a hexagon inscribed in any conic. ■

20.3. Dual Statement

Theorem 20.5 (Brianchon's Theorem). *Let a hexagon be circumscribed about a conic. Then its three main diagonals are concurrent.*

Proof for a circle. Let the hexagon be denoted by

$$A_1 A_2 A_3 A_4 A_5 A_6,$$

and suppose that its sides are tangent to a circle Γ .

Let P_i be the point of contact of the side $A_i A_{i+1}$ with Γ , where the indices are taken modulo 6. Thus

$$P_1, P_2, P_3, P_4, P_5, P_6$$

lie on the circle Γ .

By Pascal's theorem applied to the hexagon

$$P_1 P_2 P_3 P_4 P_5 P_6,$$

the three points

$$X = P_1 P_2 \cap P_4 P_5, \quad Y = P_2 P_3 \cap P_5 P_6, \quad Z = P_3 P_4 \cap P_6 P_1$$

are collinear.

We now use the polarity with respect to the circle Γ . Since $A_i A_{i+1}$ is tangent to Γ at P_i , the point P_i is the pole of the tangent line $A_i A_{i+1}$.

Hence the line $P_i P_{i+1}$ is the polar of the point A_{i+1} , because A_{i+1} is the intersection of the two tangents at P_i and P_{i+1} .

Therefore, the polar of

$$X = P_1 P_2 \cap P_4 P_5$$

is the line joining the poles of $P_1 P_2$ and $P_4 P_5$, namely

$$A_2 A_5.$$

Similarly, the polars of Y and Z are respectively

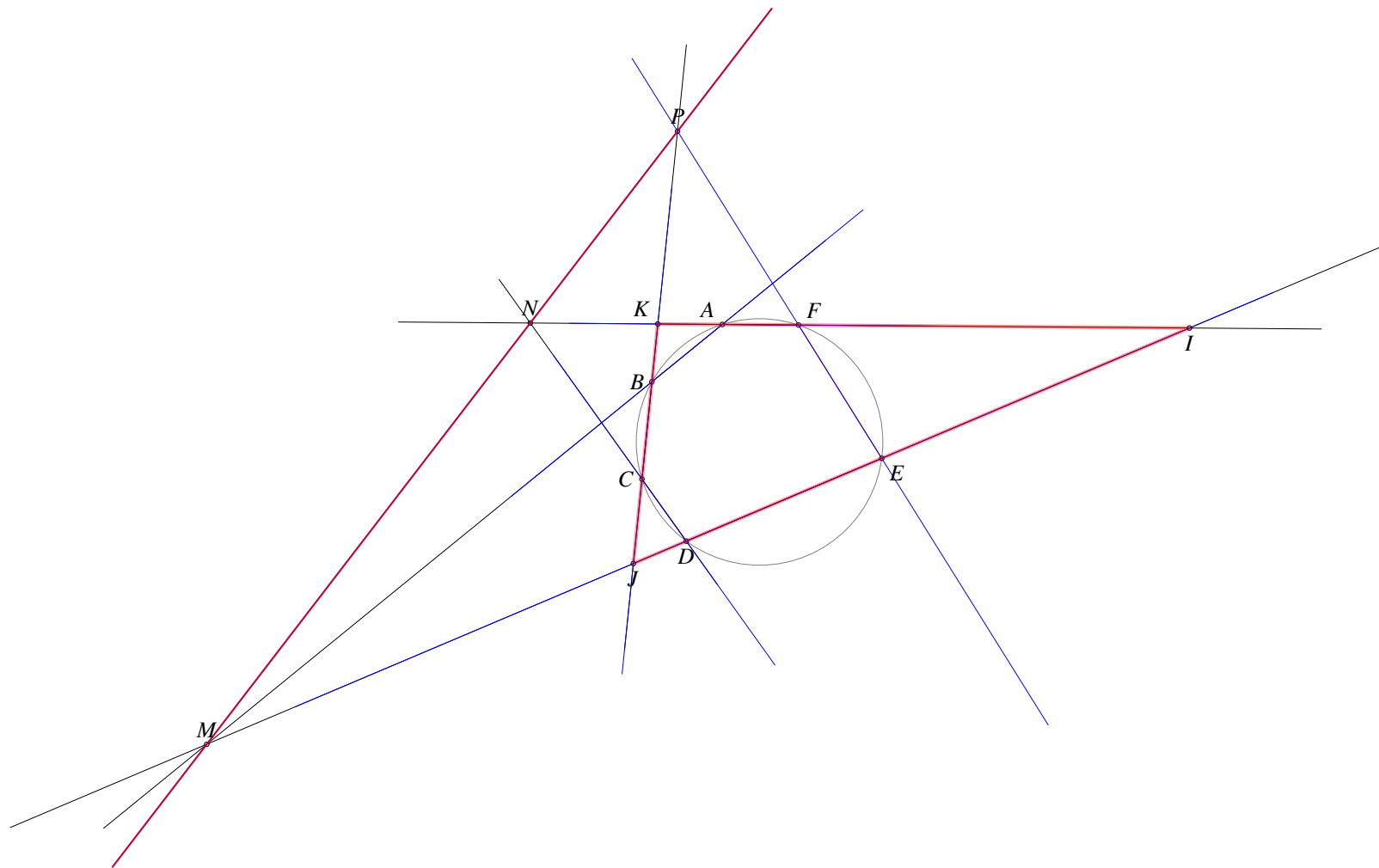
$$A_3 A_6 \quad \text{and} \quad A_4 A_1.$$

Since X, Y, Z are collinear, their polars are concurrent. Hence the three lines

$$A_2 A_5, \quad A_3 A_6, \quad A_4 A_1$$

are concurrent.

These are precisely the three main diagonals of the hexagon. Thus Brianchon's theorem holds for a hexagon circumscribed about a circle. ■



Proof. We use directed lengths.

We want to prove that the points M , N , and P are collinear. By Menelaus' theorem in the triangle IJK , this is equivalent to proving that

$$\Pi = \frac{\overline{MI}}{\overline{MJ}} \frac{\overline{PJ}}{\overline{PK}} \frac{\overline{NK}}{\overline{NI}} = 1.$$

Apply Menelaus' theorem in the triangle IJK to the line through M , A , B . Since M , A , B are collinear, we obtain

$$\frac{\overline{MI}}{\overline{MJ}} \frac{\overline{BJ}}{\overline{BK}} \frac{\overline{AK}}{\overline{AI}} = 1.$$

Hence

$$\frac{\overline{MI}}{\overline{MJ}} = \frac{\overline{BK}}{\overline{BJ}} \frac{\overline{AI}}{\overline{AK}}.$$

Similarly, applying Menelaus' theorem to the lines through P , E , F and through N , D , C , we get

$$\frac{\overline{PJ}}{\overline{PK}} = \frac{\overline{EI}}{\overline{EJ}} \frac{\overline{FI}}{\overline{FK}},$$

and

$$\frac{\overline{NK}}{\overline{NI}} = \frac{\overline{CK}}{\overline{CJ}} \frac{\overline{DJ}}{\overline{DI}}.$$

Therefore

$$\Pi = \left(\frac{\overline{BK}}{\overline{BJ}} \frac{\overline{AI}}{\overline{AK}} \right) \left(\frac{\overline{EI}}{\overline{EJ}} \frac{\overline{FI}}{\overline{FK}} \right) \left(\frac{\overline{CK}}{\overline{CJ}} \frac{\overline{DJ}}{\overline{DI}} \right).$$

Now use the power of the points I , J , and K with respect to the circle. Since the secants through these points meet the circle in the corresponding pairs of points, we have

$$\overline{AI} \overline{FI} = \overline{EI} \overline{DI},$$

$$\overline{EJ} \overline{DJ} = \overline{BJ} \overline{CJ},$$

and

$$\overline{BK} \overline{CK} = \overline{AK} \overline{FK}.$$

Substituting these three relations in the expression for Π , all factors cancel, and we obtain

$$\Pi = 1.$$

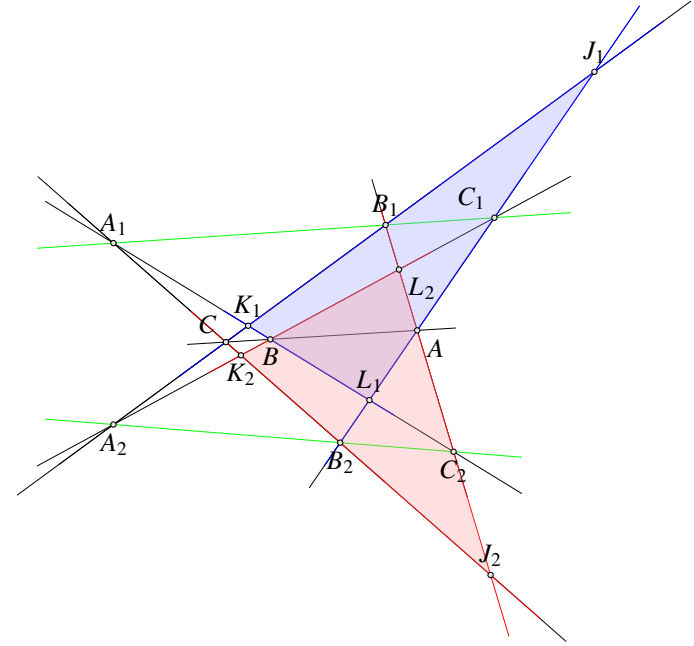
By Menelaus' theorem, the points M , N , and P are collinear. This line is called the Pascal line of the hexagon. ■

20.4. Degenerate Case

Theorem 20.6 (Pappus' Theorem). *Let A , B , C be three points on a line and A' , B' , C' three points on another line. Then the points*

$$AB' \cap A'B, \quad AC' \cap A'C, \quad BC' \cap B'C$$

are collinear.



Proof. [Affine proof using Menelaus' theorem]

The proof comes from: [[Wikipedia](#).]

Here we assume, as in the figure, that the two triples of distinct points (A_1, B_1, C_1) and (A_2, B_2, C_2) are each collinear on two distinct lines.

We further assume that the lines (B_2C_1) and (C_2B_1) meet at A , that (A_2C_1) and (C_2A_1) meet at B , and that (A_2B_1) and (B_2A_1) meet at C .

We now give a direct affine proof, under additional assumptions:

$$(A_2B_1) \cap (B_2C_1) = J_1, \quad (B_2C_1) \cap (A_1C_2) = L_1, \quad (A_2B_1) \cap (A_1C_2) = K_1.$$

The three points J_1, K_1, L_1 are distinct and non-collinear, and define a triangle (in blue in the figure).

- The line (A_1C_1) meets the three sides of the triangle at A_1, B_1, C_1 .
- The line (A_2C_2) meets the three sides of the triangle at A_2, B_2, C_2 .
- The line (B_1C_2) meets the three sides at B_1, A, C_2 .
- The line (A_2C_1) meets the three sides at A_2, B, C_1 .
- The line (A_1B_2) meets the three sides at A_1, C, B_2 .

By Menelaus' theorem, these alignments yield the following equalities:

$$\frac{\overline{A_1K_1}}{\overline{A_1L_1}} \cdot \frac{\overline{B_1J_1}}{\overline{B_1K_1}} \cdot \frac{\overline{C_1L_1}}{\overline{C_1J_1}} = 1,$$

$$\frac{\overline{A_2J_1}}{\overline{A_2K_1}} \cdot \frac{\overline{B_2L_1}}{\overline{B_2J_1}} \cdot \frac{\overline{C_2K_1}}{\overline{C_2L_1}} = 1,$$

$$\frac{\overline{B_1K_1}}{\overline{B_1J_1}} \cdot \frac{\overline{AJ_1}}{\overline{AL_1}} \cdot \frac{\overline{C_2L_1}}{\overline{C_2K_1}} = 1,$$

$$\frac{\overline{A_2K_1}}{\overline{A_2J_1}} \cdot \frac{\overline{BL_1}}{\overline{BK_1}} \cdot \frac{\overline{C_1J_1}}{\overline{C_1L_1}} = 1,$$

$$\frac{\overline{A_1L_1}}{\overline{A_1K_1}} \cdot \frac{\overline{CK_1}}{\overline{CJ_1}} \cdot \frac{\overline{B_2J_1}}{\overline{B_2L_1}} = 1.$$

Multiplying these five equalities and simplifying, we obtain:

$$\frac{\overline{AJ_1}}{\overline{AL_1}} \cdot \frac{\overline{BL_1}}{\overline{BK_1}} \cdot \frac{\overline{CK_1}}{\overline{CJ_1}} = 1.$$

By the converse of Menelaus' theorem, this proves that the three points A , B , and C are collinear. A similar proof can be obtained by modifying the additional assumptions, using the triangle $J_2K_2L_2$ (in red in the figure). In that case, the three lines (B_1C_2) , (A_2C_1) , and (A_1B_2) exchange their roles with the three lines (B_2C_1) , (A_1C_2) , and (A_2B_1) .

■

20.5. Application: Pascal' theorem weak version

The statement and proof are from the site :

Theorem 20.7 (Pascal' theorem). *Given 6 points (which can be coincident) on the circumference of a circle labelled A, C, E, B, F and D in that order around the circle, the intersections P, Q and R of (AB) and (DE) , (AF) and (CD) , and (BC) and (EF) are collinear.*

Proof. This demonstration is based on a general case. In some cases, points may be confused and some lines may be parallel, so for these particular cases, the proof must be adapted.

Let Q be the intersection of (CD) and (AF) , let P be the intersection of (AB) and (DF) , and let R be the intersection of (BC) and (EF) . We will prove these three points are collinear.

Let U be the intersection of (CD) and (EF) , let V be the intersection of (AB) and (EF) , and let W be the intersection of (AB) and (CD) . By Menelaus in the triangle UVW and the transversal (CD) , we have $\frac{VQ}{UQ} \cdot \frac{UC}{WC} \cdot \frac{WD}{VD} = 1$.

Then with the transversal (AB) , we have $\frac{VA}{UA} \cdot \frac{UB}{WB} \cdot \frac{WP}{VP} = 1$.

Finally with the transversal (EF) , we get $\frac{VF}{UF} \cdot \frac{UR}{WR} \cdot \frac{WE}{VE} = 1$.

Multiply the three identities and rearrange the terms:

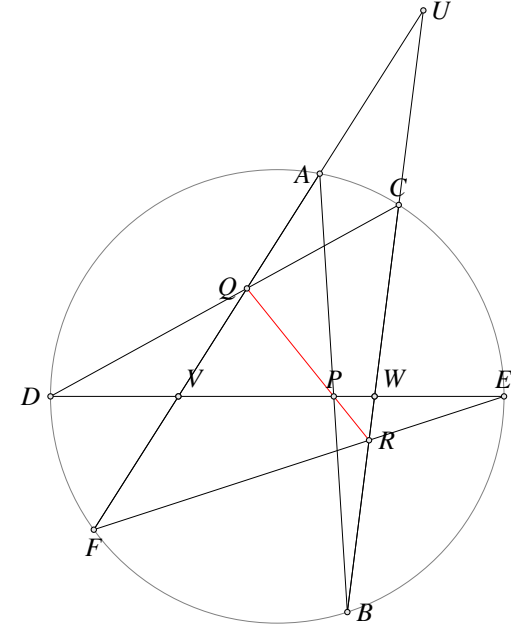
$$\frac{VQ}{UQ} \cdot \frac{UR}{WR} \cdot \frac{WP}{VP} = 1.$$

■

File `pascal.lua`:

```
init_elements()
z.O = point(0, 0)
z.E = point(3, 0)
C.OE = circle(z.O, z.E)
z.C = C.OE:point(0.16)
z.A = C.OE:point(0.22)
z.D = C.OE:point(0.5)
z.F = C.OE:point(0.6)
z.B = C.OE:point(0.8)
L.AB = line(z.A, z.B)
L.DE = line(z.D, z.E)
```

```
L.AF = line(z.A, z.F)
L.CD = line(z.C, z.D)
L.BC = line(z.B, z.C)
L.EF = line(z.E, z.F)
z.P = intersection(L.AB, L.DE)
z.Q = intersection(L.AF, L.CD)
z.R = intersection(L.BC, L.EF)
z.U = intersection(L.AF, L.BC)
z.V = intersection(L.AF, L.DE)
z.W = intersection(L.BC, L.DE)
```



Another demonstration

Proof. Let's call Ω the circle passing through A, B, C, D, E and F . Projecting through F onto Ω .
 $(J, K; Q, R) = (J, K; A, E)$
 Projecting through C onto Ω .
 $(J, K; Q, R) = (J, K; D, B)$
 Projecting through B onto (J, K) .
 $(J, K; A, E) = (J, K; (AB) \cap (JK), (EB) \cap (JK))$
 Projecting through E onto (J, K) .
 $(J, K; D, B) = (J, K; (DE) \cap (JK), (BE) \cap (JK))$
 $(EB) \cap (JK) = (BE) \cap (JK)$ and $(AB) \cap (JK) = (DE) \cap (JK) = P$
 Finally P, Q, R are collinear.

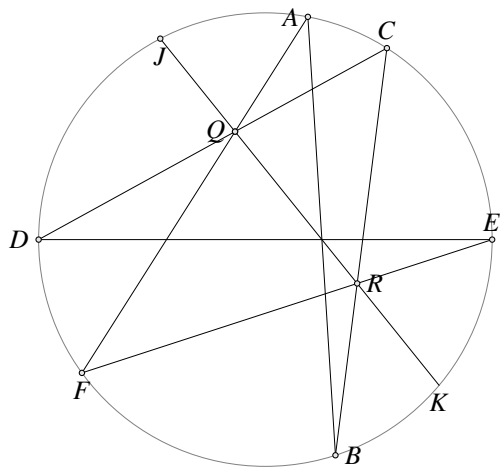
■

File `pascal_proj.lua`:

```

init_elements()
z.O = point(0, 0)
z.E = point(3, 0)
C.OE = circle(z.O, z.E)
z.C = C.OE:point(0.16)
z.A = C.OE:point(0.22)
z.D = C.OE:point(0.5)
z.F = C.OE:point(0.6)
z.B = C.OE:point(0.8)
L.AB = line(z.A, z.B)
L.DE = line(z.D, z.E)
L.AF = line(z.A, z.F)
L.CD = line(z.C, z.D)
L.BC = line(z.B, z.C)
L.EF = line(z.E, z.F)
z.P = intersection(L.AB, L.DE)
z.Q = intersection(L.AF, L.CD)
z.R = intersection(L.BC, L.EF)
L.QR = line(z.Q, z.R)
z.J, z.K = intersection(C.OE, L.QR)

```



20.6. Pascal's theorem with a triangle

File pascal_tr.lua:

```

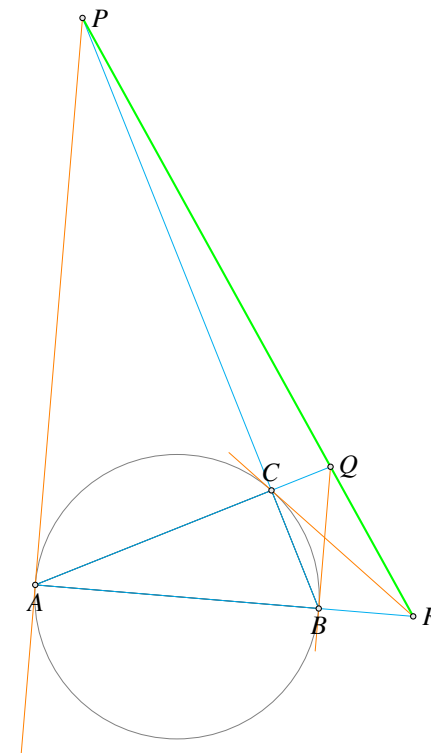
init_elements()
z.A = point(0, 0)
z.B = point(3, -0.25)
z.C = point(2.5, 1)
T.ABC = triangle(z.A, z.B, z.C)
z.O = T.ABC.circumcenter
C.OA = circle(z.O, z.A)
L.Ta = C.OA:tangent_at(z.A)

```

```

L.Tb = C.OA:tangent_at(z.B)
L.Tc = C.OA:tangent_at(z.C)
z.P = intersection(T.ABC.bc, L.Ta)
z.Q = intersection(T.ABC.ca, L.Tb)
z.R = intersection(T.ABC.ab, L.Tc)

```



20.7. Application of Pascal's theorem

Lemma 20.8 (Polar and Pole). *If through a point X there be drawn any straight line to meet the circle in A and B , the locus of the point of intersection of tangents at A and B is called the polar of X ; also X is called the pole of the polar.*

Proof. The triangles OXU and OMN are similar:

$$XO \times MO = UO \times NO$$

The triangles OUA and ONB are similar:

$$UO \times NO = AO \times BO$$

$$\text{Therefore, } XO \times MO = AO^2$$

■

File pole.lua:

```

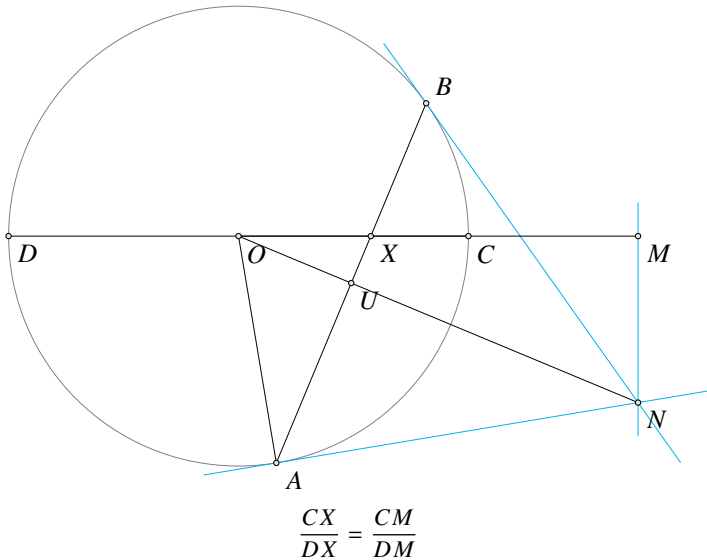
init_elements()
z.O = point(0, 0)
z.A = point(0.5, -3)
C.OA = circle(z.O, z.A)

```

```

z.X = point(1.75, 0)
L.OX = line(z.O, z.X)
L.RS = line(L.OX.north_pb, L.OX.south_pb)
z.R, z.S = intersection(C.OA, L.RS)
L.AX = line(z.A, z.X)
L.TA = C.OA:tangent_at(z.A)
z.Ax, z.Ay = L.TA:get()
z.B = intersection(L.AX, C.OA)
L.TB = C.OA:tangent_at(z.B)
z.Bx, z.By = L.TB:get()
L.TR = C.OA:tangent_at(z.R)
L.TS = C.OA:tangent_at(z.S)
z.Rx, z.Ry = L.TR:get()
z.Sx, z.Sy = L.TS:get()
z.M = intersection(L.TR, L.TS)
z.N = intersection(L.TA, L.TB)
z.U = tkz.midpoint(z.A, z.B)
z.C, z.D = intersection(C.OA, L.OX)

```



Exercise 20.9 (Polar and Pascal). Given a cyclic quadrilateral $ABCD$ if the intersection of (AC) and (BD) is P , the intersection of (AB) and (CD) is Q and the intersection of (AD) and (BC) is R , prove the polar of P with respect of $C(O, A)$ passes through Q and R .

File polar.lua:

```

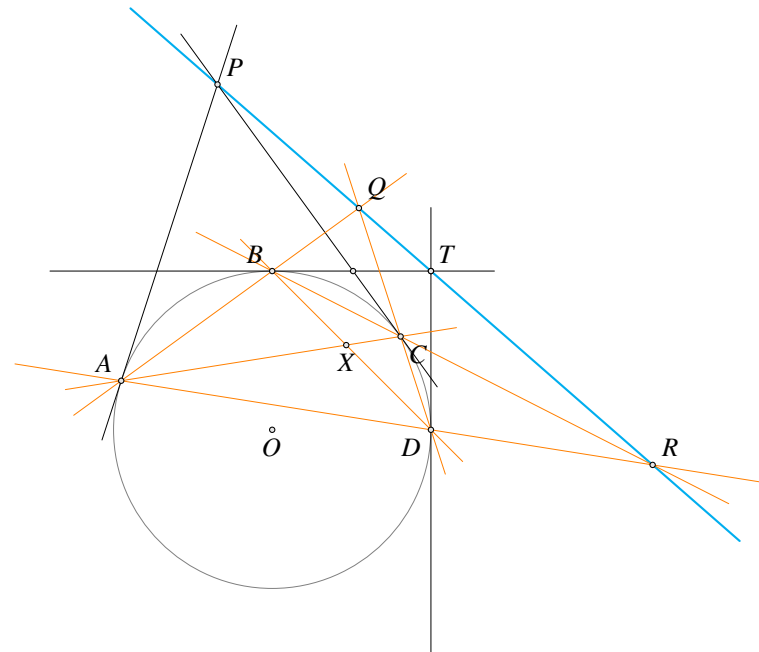
init_elements()
z.O = point(0, 0)
z.D = point(3, 0)
C.OD = circle(z.O, z.D)
z.B = C.OD:point(0.25)
z.A = C.OD:point(0.45)

```

```

z.C = C.OD:point(0.10)
L.AC = line(z.A, z.C)
L.AB = line(z.A, z.B)
L.CD = line(z.C, z.D)
L.AD = line(z.A, z.D)
L.BD = line(z.B, z.D)
L.BC = line(z.B, z.C)
z.X = intersection(L.AC, L.BD)
z.Q = intersection(L.AB, L.CD)
z.R = intersection(L.AD, L.BC)
L.QR = line(z.Q, z.R)
L.Ta = C.OD:tangent_at(z.A)
L.Tb = C.OD:tangent_at(z.B)
L.Tc = C.OD:tangent_at(z.C)
L.Td = C.OD:tangent_at(z.D)
z.Ax, z.Ay = L.Ta:get()
z.Bx, z.By = L.Tb:get()
z.Cx, z.Cy = L.Tc:get()
z.Dx, z.Dy = L.Td:get()
z.Ibd = intersection(L.Tb, L.Td)
z.P = intersection(L.Ta, L.Tc)
z.T = intersection(L.Tb, L.Td)
z.Ibc = intersection(L.Tb, L.Tc)

```



Proof. By the Pascal's Theorem on $AACBBD$, we have P, Q, R collinear. Then with hexagon $ABBCCD$, we have Q, R, T collinear. Finally, P, Q, R, T are collinear.

T_a and T_c are tangent at the circle at A and C , the intersection X ($AC \cap BD$) lies on the polar of P , i.e. (AC) (Refer to previous Lemma). Similarly, X lies on the polar of T , which leads to P and T belonging at the polar of X , so (PT) is the polar of X . Since P, Q, R and T are collinear, then Q and R are also on the polar of X . RQ is the polar of X . ■

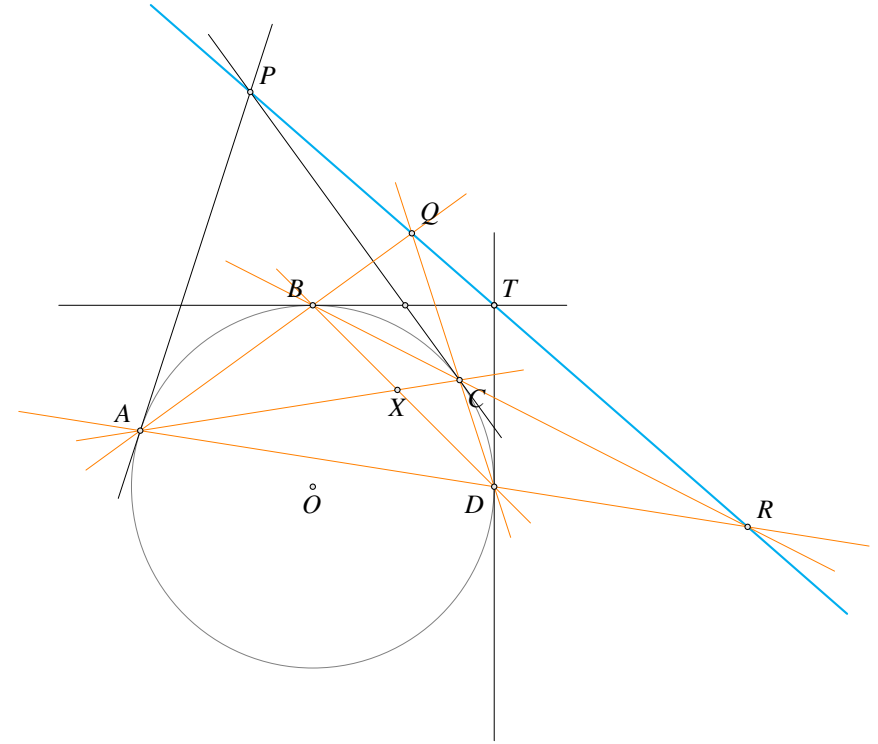
21. Poles and Polars

21.1. Polar and Pascal's theorem

Example 21.1 (Polar and Pascal's theorem). Given a cyclic quadrilateral $ABCD$ if the intersection of (AC) and (BD) is P , the intersection of (AB) and (CD) is Q and the intersection of (AD) and (BC) is R , prove the polar of P with respect of $C(O, A)$ passes through Q and R .

File polar.lua:

```
init_elements()
z.O = point(0, 0)
z.D = point(3, 0)
C.OD = circle(z.O, z.D)
z.B = C.OD:point(0.25)
z.A = C.OD:point(0.45)
z.C = C.OD:point(0.10)
L.AC = line(z.A, z.C)
L.AB = line(z.A, z.B)
L.CD = line(z.C, z.D)
L.AD = line(z.A, z.D)
L.BD = line(z.B, z.D)
L.BC = line(z.B, z.C)
z.X = intersection(L.AC, L.BD)
z.Q = intersection(L.AB, L.CD)
z.R = intersection(L.AD, L.BC)
L.QR = line(z.Q, z.R)
L.Ta = C.OD:tangent_at(z.A)
L.Tb = C.OD:tangent_at(z.B)
L.Tc = C.OD:tangent_at(z.C)
L.Td = C.OD:tangent_at(z.D)
z.Ax, z.Ay = L.Ta:get()
z.Bx, z.By = L.Tb:get()
z.Cx, z.Cy = L.Tc:get()
z.Dx, z.Dy = L.Td:get()
z.Ibd = intersection(L.Tb, L.Td)
z.P = intersection(L.Ta, L.Tc)
z.T = intersection(L.Tb, L.Td)
z.Ibc = intersection(L.Tb, L.Tc)
```

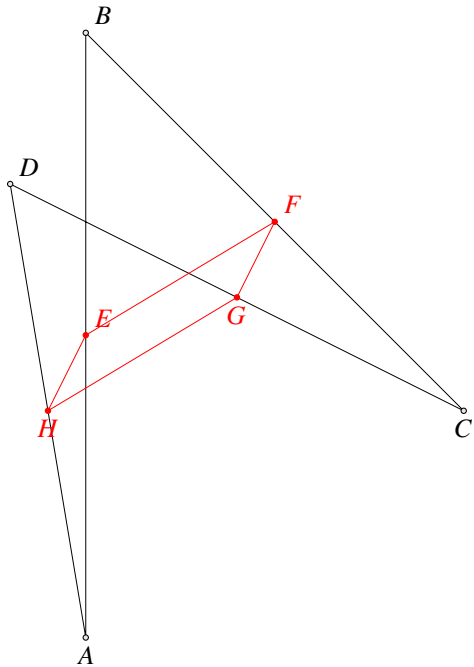


Part 7. Polygons

22. Parallelograms

22.1. Varignon's Theorem

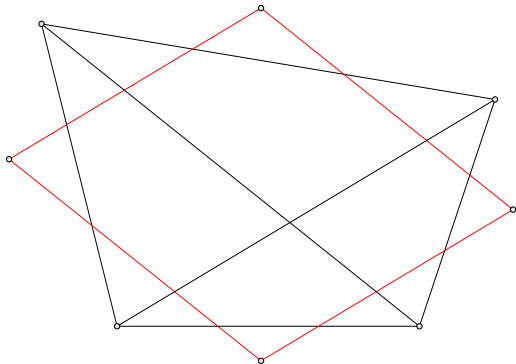
Theorem 22.1 (Varignon's Theorem). *In Euclidean geometry, Varignon's theorem holds that the midpoints of the sides of an arbitrary quadrilateral form a parallelogram, called the Varignon parallelogram. It is named after Pierre Varignon, whose proof was published posthumously in 1731. [Wikipedia]*



22.2. Wittenbauer's Parallelogram

Theorem 22.2 (Wittenbauer's Parallelogram). *Divide the sides of a quadrilateral into three equal parts. The figure formed by connecting and extending adjacent points on either side of a polygon vertex is a parallelogram known as Wittenbauer's parallelogram.*

Weisstein, Eric W. "Wittenbauer's Parallelogram." From MathWorld—A Wolfram Web Resource.

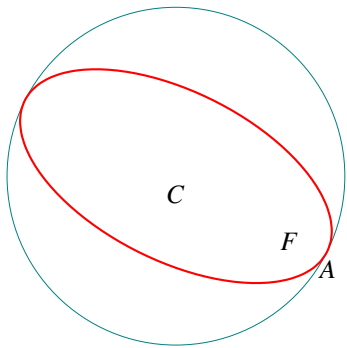


Part 8. Conics

23. Ellipse

23.1. Ellipse

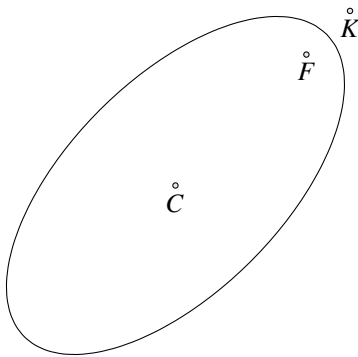
File ellipse.lua:



23.2. Ellipse with radii

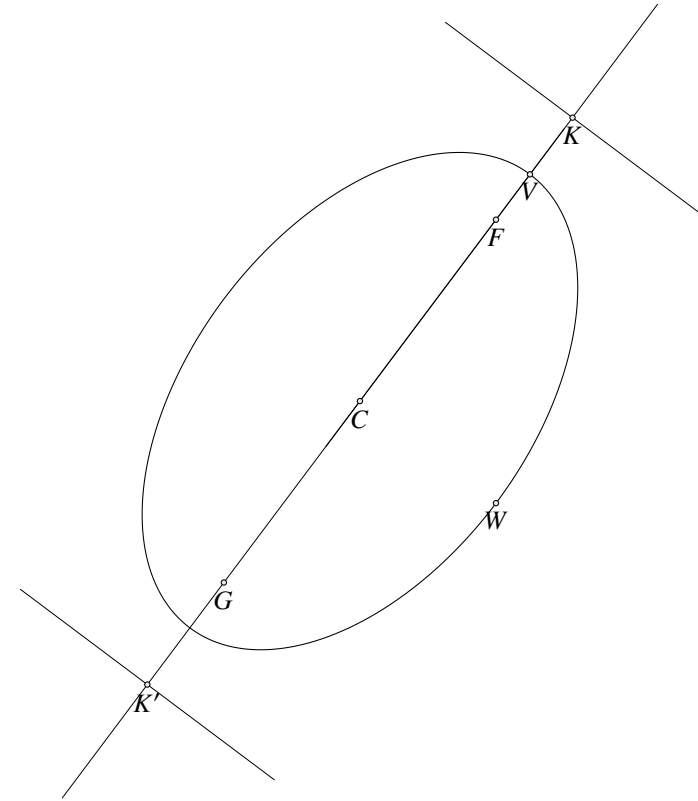
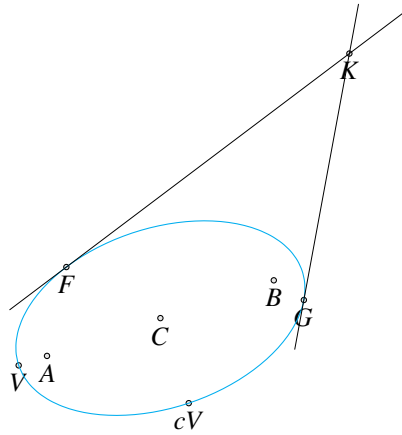
Example 23.1 (Ellipse with radii). In this example, K is the projection of the focus F on the directrix.

File radii.lua:



23.3. Ellipse_with_foci

File focii.lua:



23.4. Ellipse with Center, Vertex, and Covertex

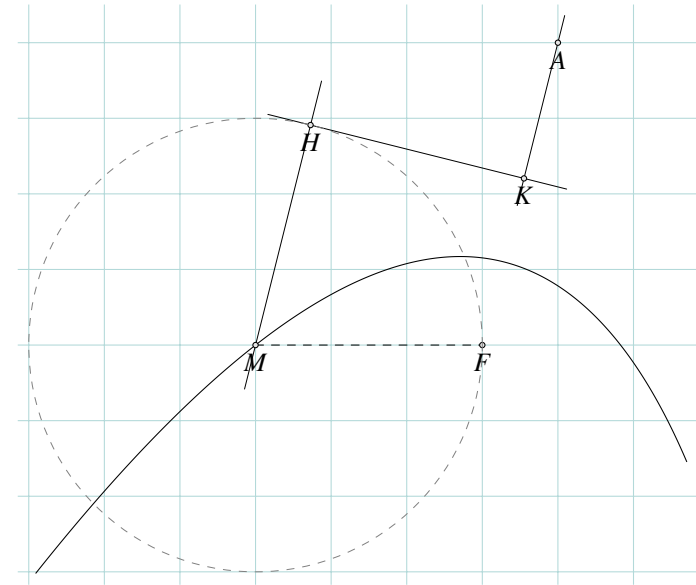
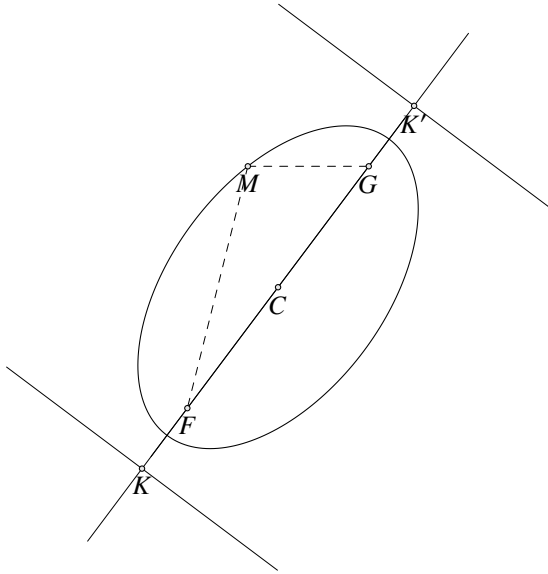
This construction defines an ellipse from its center, a vertex, and a covertex. From these three points, the semi-axes lengths a and b are computed, allowing the eccentricity e to be derived. The foci and the direction of the directrix are then constructed accordingly, making it possible to define the ellipse using the `conic` method.

File `ellipse2.lua`:

23.5. Ellipse with Foci and a Point

This construction defines an ellipse given its two foci and a single point on the curve. The fundamental property $MF + MG = 2a$, where F and G are the foci and M a point on the ellipse, allows the computation of the semi-major axis a . From this, the eccentricity and the direction of the directrix can be derived.

File `ellipse3.lua`:



24. Parabola

24.1. Parabola with Focus, Axis of Symmetry, and a Point on the Curve

This construction illustrates how to define a parabola from its focus, its axis of symmetry, and a single point on the curve. The axis is determined as the line orthogonal to the segment joining the focus to the given point, passing through a specific intersection derived from a circle construction. This method is particularly useful when the directrix is not known explicitly.

File `parabola1.lua`:

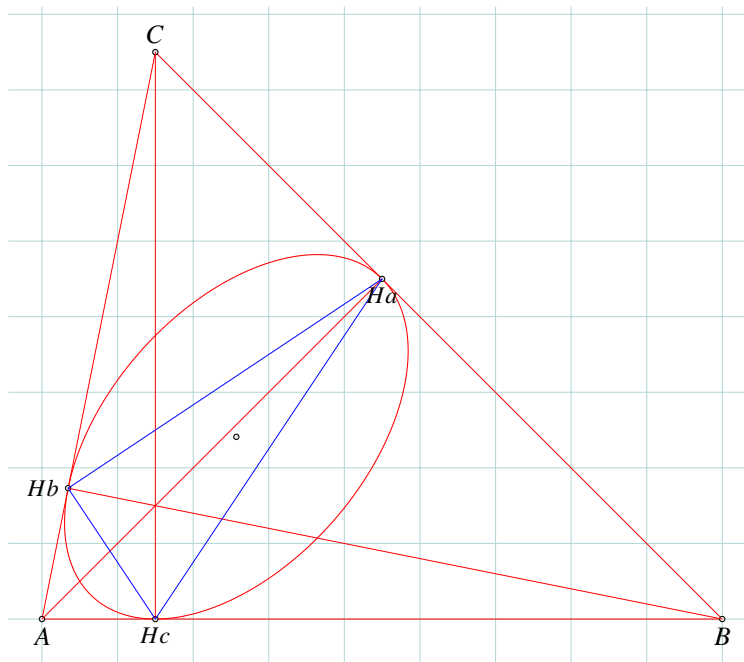
25. Conics in a Triangle

25.1. Orthic_inellipse with `search_ellipse`

Definition 25.1. The orthic inconic of a triangle is the inconic with inconic parameters $x : y : z = \cos(A) : \cos(B) : \cos(C)$.

It is an ellipse for acute triangles and a hyperbola for obtuse triangles.

File `inellipse.lua`:



25.2. Kiepert hyperbola

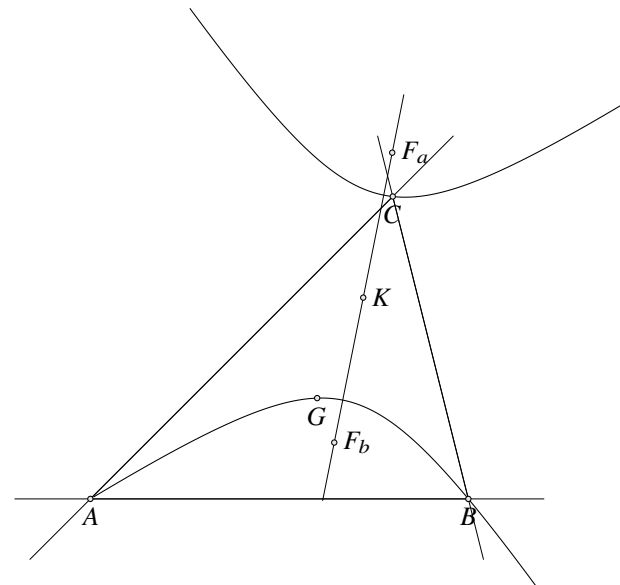
Definition 25.2. Kiepert hyperbola In triangle geometry, the Kiepert conics are two special conics associated with the reference triangle. One of them is a hyperbola, called the Kiepert hyperbola and the other is a parabola, called the Kiepert parabola.

It has been proved that the Kiepert hyperbola is the hyperbola passing through the vertices, the centroid and the orthocenter of the reference triangle [Wikipedia]

File kiepert_hyperbola.lua:

```
init_elements()
z.A = point(0, 0)
z.B = point(5, 0)
z.C = point(4, 4)
T = triangle(z.A, z.B, z.C)
z.K = T:kimberling(115)
z.circumcenter = T.circumcenter
z.G = T.centroid
L.brocard = T:brocard_axis()
C.circum = circle(z.circumcenter, z.A)
z.M, z.N = intersection(L.brocard, C.circum)
L.asx = T:simson_line(z.M)
L.asy = T:simson_line(z.N)
z.ux, z.uy = L.asx:get()
z.vx, z.vy = L.asy:get()
HY = T:kiepert_hyperbola()
curve = HY:points(-3, 3, 50)
```

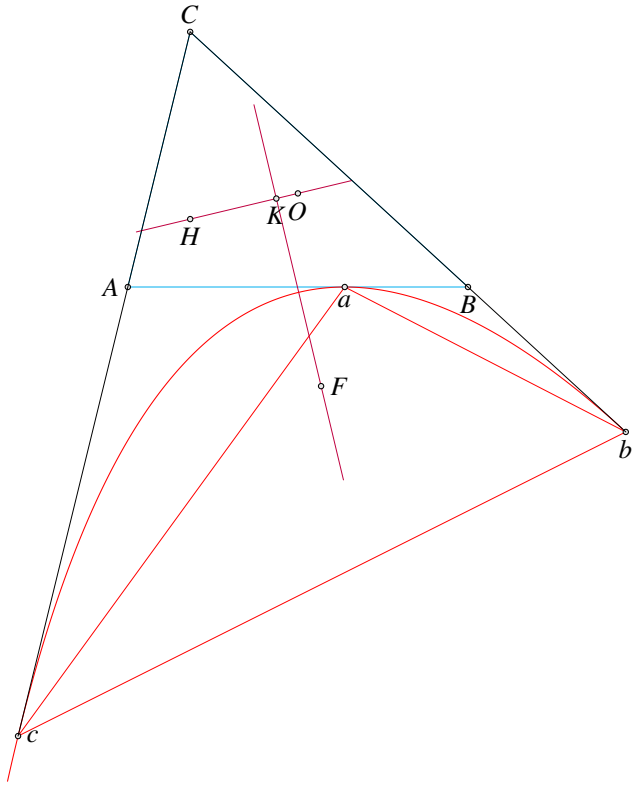
```
curveb = HY:points(-3, 3, 50, "swap")
z.F_a, z.F_b = HY.Fa, HY.Fb
```



25.3. Kiepert parabola

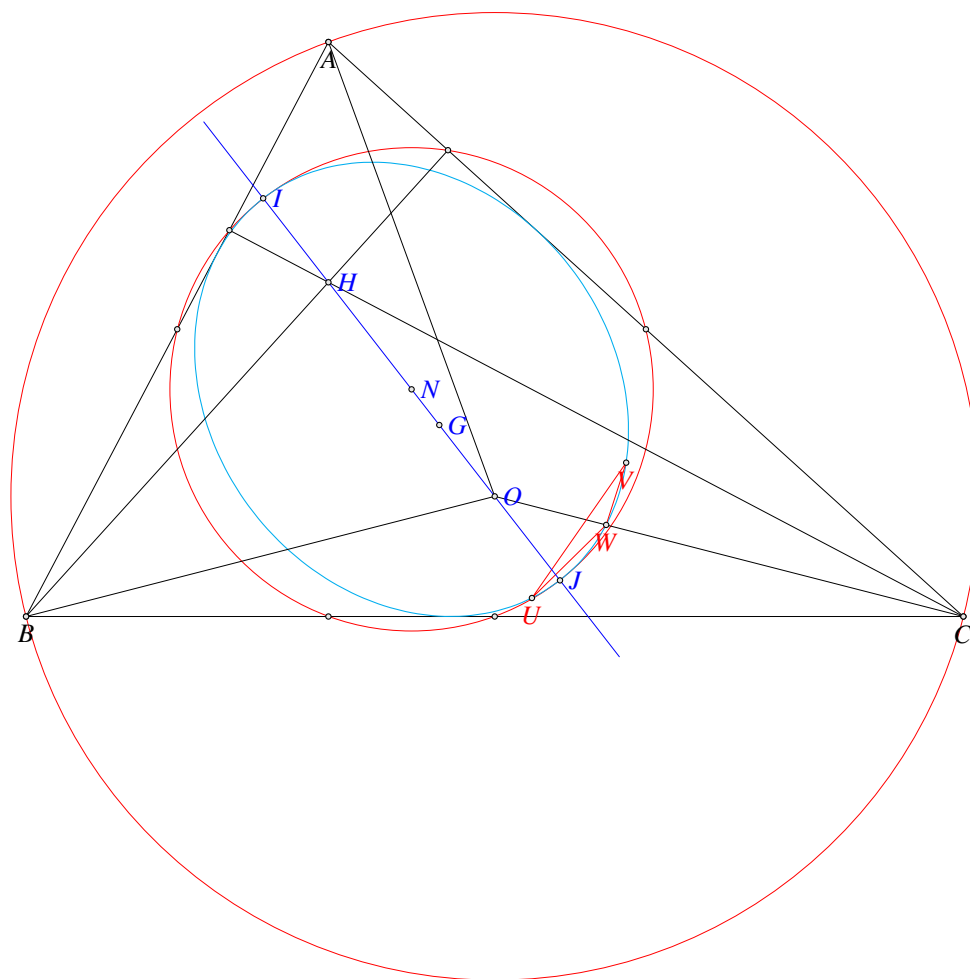
Definition 25.3. Kiepert parabola The Euler line of a triangle is the conic section directrix of the Kiepert parabola. In fact, the directrices of all parabolas inscribed in a triangle pass through the orthocenter. The triangle formed by the points of contact is called the Steiner triangle. The Kiepert parabola is tangent to the sides of the triangle (or their extensions), the line at infinity, and the Lemoine axis. The focus of the parabola has is Kimberling center X_{110} .

File kiepert_parabola.lua:



Definition 25.4. Euler ellipse Euler's ellipse is a conic section, tangent to the three sides of a triangle, with foci at the orthocentre and the centre of the circumscribed circle.

Let ABC be an acute-angled triangle, neither right-angled nor equilateral.



Part 9. Advanced Theorems

26. Feuerbach Theorem

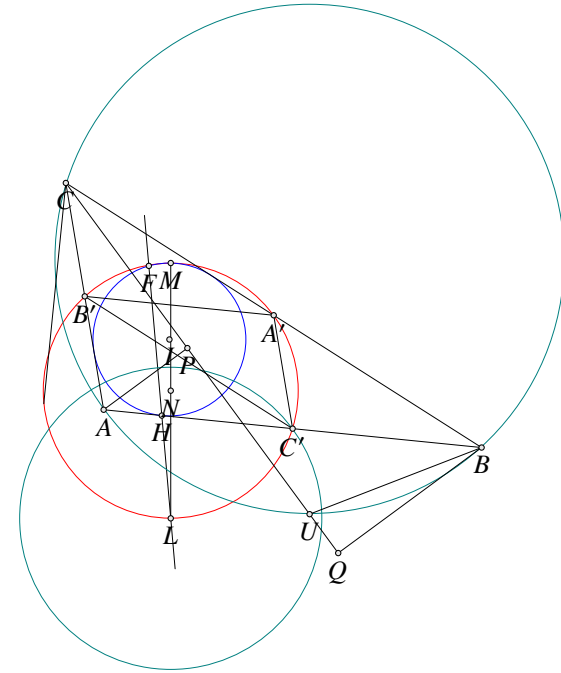
26.1. Feuerbach theorem

File feuerbach.lua:

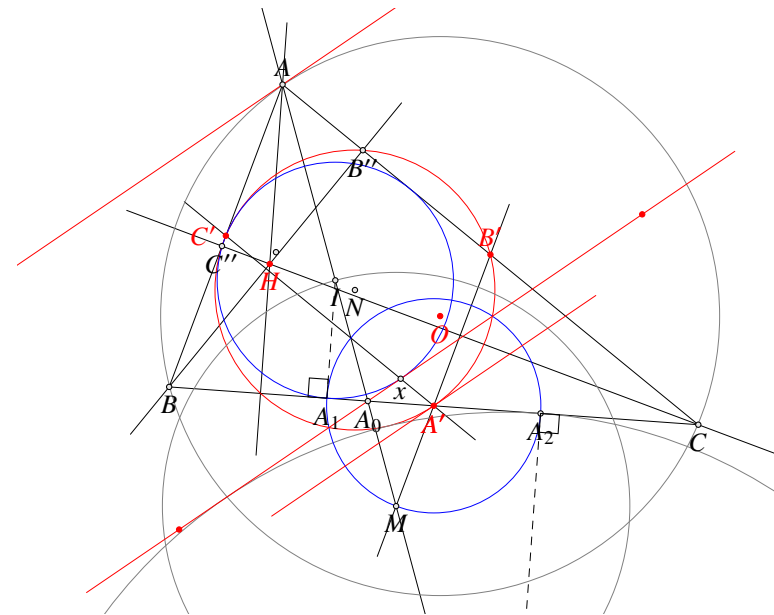
```

init_elements()
z.A = point(0, 0)
z.B = point(5, -0.5)
z.C = point(-0.5, 3)
T.ABC = triangle(z.A, z.B, z.C)
z.O = T.ABC.circumcenter
z.N = T.ABC.eulercenter
z.I, z.K = T.ABC.in_circle():get()
z.H = T.ABC.ab:projection(z.I)
z.Ap, z.Bp, z.Cp = T.ABC.medial():get()
C.IH = circle(z.I, z.H)
C.NAp = circle(z.N, z.Ap)
C.OA = circle(z.O, z.A)
z.U = C.OA.south
z.L = C.NAp.south
z.M = C.NAp.north
z.X = T.ABC.ab:projection(z.C)
L.CU = line(z.C, z.U)
L.ML = line(z.M, z.L)
z.P = L.CU:projection(z.A)
z.Q = L.CU:projection(z.B)
L.LH = line(z.L, z.H)
z.F = intersection(L.LH, C.IH)

```



26.2. Feuerbach_Axel



Soit ABC un triangle.
On note (Ω) le cercle d'Euler,
 $A'B'C'$ les milieux des côtés,

$A''B''C''$ les pieds des hauteurs,
 (Γ) le cercle inscrit, I son centre,
 $(\Gamma_a), (\Gamma_b), (\Gamma_c)$ les cercles exinscrits dans les angles $\widehat{A}, \widehat{B}, \widehat{C}$,
 I_a, I_b, I_c leurs centres,
 A_1, B_1, C_1 les points de contact de (Γ) avec les côtés,
 A_2, B_2, C_2 ceux de $(\Gamma_a), (\Gamma_b), (\Gamma_c)$.

(1) **Première question préliminaire**

Soit l'inversion de pôle O et de puissance k^2 , (C) le cercle de centre O et de rayon k , (Γ') un cercle orthogonal à (C) . Montrer que $i(\Gamma') = (\Gamma')$.

Proof. Soit (Γ') un cercle orthogonal au cercle d'inversion (C) , de centre O et de rayon k . Soit $M \in (\Gamma')$ et $M' = i(M)$.

Comme (Γ') est orthogonal à (C) , la puissance de O par rapport à (Γ') vaut

$$\mathcal{P}_{(\Gamma')}(O) = k^2.$$

Par conséquent, si la droite OM recoupe (Γ') en un second point M' , on a

$$OM \cdot OM' = \mathcal{P}_{(\Gamma')}(O) = k^2.$$

Mais c'est précisément la relation caractéristique de l'inversion de centre O et de puissance k^2 :

$$OM \cdot i(M) = k^2.$$

Donc le second point d'intersection de la droite OM avec (Γ') est l'image de M par l'inversion. Ainsi $i(M) \in (\Gamma')$.

Comme ceci vaut pour tout point M de (Γ') , le cercle (Γ') est globalement invariant :

$$i(\Gamma') = (\Gamma').$$

■

(2) **Deuxième question préliminaire**

Montrer que la tangente T' en A' au cercle d'Euler d'un triangle ABC est parallèle à la tangente en A au cercle circonscrit à ce même triangle.

Proof. Notons A', B', C' les milieux respectifs de $[BC]$, $[CA]$ et $[AB]$. Le cercle d'Euler du triangle ABC est le cercle circonscrit du triangle médial $A'B'C'$.

Soit T' la tangente en A' au cercle d'Euler, et T la tangente en A au cercle circonscrit de ABC .

Comme $A'C'$ est une corde du cercle d'Euler, le théorème de l'angle formé par une tangente et une corde donne

$$(T', A'C') = \widehat{A'B'C'} \pmod{\pi}.$$

Or, dans le triangle médial, on a

$$A'C' \parallel AC \quad \text{et} \quad B'A' \parallel AB, \quad B'C' \parallel BC.$$

Ainsi

$$\widehat{A'B'C'} = \widehat{ABC}.$$

Donc

$$(T', A'C') = \widehat{ABC} \pmod{\pi}.$$

D'autre part, dans le cercle circonscrit à ABC , la corde AC donne, par le même théorème,

$$(T, AC) = \widehat{ABC} \pmod{\pi}.$$

Comme $A'C' \parallel AC$, on en déduit

$$(T', AC) = (T', A'C') = \widehat{ABC} = (T, AC) \pmod{\pi}.$$

Par conséquent,

$$T' \parallel T.$$

La tangente en A' au cercle d'Euler est donc parallèle à la tangente en A au cercle circonscrit. ■

(3) **Troisième question préliminaire**

Soit ABC un triangle, AI la bissectrice intérieure de $\widehat{A}$ et T la tangente en A au cercle circonscrit. Montrer que

$$(AI, BC) = (AT, AI) \pmod{\pi}.$$

En déduire que

$$(AI, BC) = (T', AI) \pmod{\pi}.$$

Proof. Par le théorème tangente-corde,

$$(AT, AB) = \widehat{ACB}.$$

Comme AI est la bissectrice de $\widehat{A}$,

$$(AB, AI) = \frac{1}{2} \widehat{A}.$$

Donc

$$(AT, AI) = \widehat{ACB} + \frac{1}{2} \widehat{A}.$$

Or

$$\widehat{ACB} = \pi - \widehat{A} - \widehat{ABC},$$

d'où

$$(AT, AI) = \pi - \widehat{ABC} - \frac{1}{2} \widehat{A}.$$

De même,

$$(AI, BC) = \frac{1}{2} \widehat{A} + (\pi - \widehat{ABC}) = \pi - \widehat{ABC} - \frac{1}{2} \widehat{A}.$$

Donc

$$(AI, BC) = (AT, AI) \pmod{\pi}.$$

Comme $T' \parallel T$, on obtient

$$(AI, BC) = (T', AI) \pmod{\pi}.$$

■

(4) Montrer que $CA_1 = p - c$ et $BA_2 = p - c$ où p désigne le demi-périmètre de ABC et c la longueur de BA . En déduire que A' est le milieu de A_1A_2 .

Proof. Posons

$$a = BC, \quad b = CA, \quad c = AB, \quad p = \frac{a + b + c}{2}.$$

Si A_1 est le point de contact du cercle inscrit avec BC , alors les longueurs tangentielles donnent

$$CA_1 = p - c.$$

Si A_2 est le point de contact de l'exercerle opposé à A avec la droite BC , on a de même

$$BA_2 = p - c.$$

Donc

$$CA_1 = BA_2.$$

Or A' est le milieu de $[BC]$, donc

$$CA' = BA'.$$

Comme A_1 et A_2 sont sur la droite BC , cette égalité montre que A_1 et A_2 sont symétriques par rapport à A' . Ainsi, A' est le milieu de $[A_1A_2]$. ■

- (5) Soit i l'inversion de pôle A' et de puissance $A'H_a^2$. Quelles sont les images de (Γ) et (Γ_a) par i ? (On pourra noter (C) le cercle de diamètre A_1A_2 .)

Proof. Considérons l'inversion i de centre A' et de puissance $A'A''^2$. Soit (C) son cercle d'inversion, c'est-à-dire le cercle de centre A' et de rayon $A'A''$.

D'après la question précédente, A' est le milieu de $[A_1A_2]$. Or le cercle d'Euler passe par A' et A'' , et son centre est le milieu de $[OH]$; en particulier on a

$$A'A'' = A'A_1 = A'A_2.$$

Ainsi, le cercle (C) est aussi le cercle de diamètre A_1A_2 .

Le cercle inscrit (Γ) est tangent à la droite BC en A_1 . Donc la puissance de A' par rapport à (Γ) vaut

$$\mathcal{P}_{(\Gamma)}(A') = A'A_1^2 = A'A''^2.$$

Ainsi, (C) est orthogonal à (Γ) .

De même, le cercle exinscrit (Γ_a) est tangent à la droite BC en A_2 , donc

$$\mathcal{P}_{(\Gamma_a)}(A') = A'A_2^2 = A'A''^2.$$

Ainsi, (C) est orthogonal à (Γ_a) .

D'après la première question préliminaire, tout cercle orthogonal au cercle d'inversion est invariant par cette inversion. On en déduit

$$i(\Gamma) = \Gamma \quad \text{et} \quad i(\Gamma_a) = \Gamma_a.$$

- (6) Montrer que $(D) = i(\Omega)$ est parallèle à la tangente en A' à (Ω) . En déduire que

$$(D, AI) = (AI, BC) \pmod{\pi}.$$

Proof. Le cercle d'Euler (Ω) passe par le centre A' de l'inversion ; son image est donc une droite (D) .

Soit T' la tangente en A' à (Ω) . Plaçons l'origine en A' et l'axe des abscisses sur T' . Si N est le centre de (Ω) et $R = NA'$, alors (Ω) a pour équation

$$x^2 + y^2 - 2Ry = 0.$$

Pour un point $M = (x, y)$ de (Ω) , son image par l'inversion de puissance $k^2 = A'A''^2$ est

$$M' = \left(\frac{k^2x}{x^2 + y^2}, \frac{k^2y}{x^2 + y^2} \right).$$

Comme $x^2 + y^2 = 2Ry$, on obtient

$$M' = \left(\frac{k^2x}{2Ry}, \frac{k^2}{2R} \right).$$

Ainsi, tous les points M' ont la même ordonnée : l'image de (Ω) est donc une droite parallèle à T' .

Donc

$$(D) \parallel T'.$$

Or, d'après la question précédente,

$$(T', AI) = (AI, BC) \pmod{\pi}.$$

Par suite,

$$(D, AI) = (AI, BC) \pmod{\pi}.$$

- (7) Soit (D') la tangente intérieure commune à (Γ) et (Γ_a) autre que BC . Montrer que $(D) \parallel (D')$.

Proof. Les centres I et I_a des cercles (Γ) et (Γ_a) appartiennent à la bissectrice AI . Par conséquent, les deux tangentes intérieures communes à ces deux cercles sont symétriques par rapport à AI .

Comme BC est l'une de ces tangentes, l'autre, notée (D') , vérifie

$$(D', AI) = (AI, BC) \pmod{\pi}.$$

Mais la question précédente a donné

$$(D, AI) = (AI, BC) \pmod{\pi}.$$

Donc

$$(D, AI) = (D', AI) \pmod{\pi},$$

ce qui entraîne

$$(D) \parallel (D').$$

- (8) Soit A_0 le point d'intersection de BC et (D') . Montrer que (A', A_0, A_1, A_2) est un quaterne harmonique. Puis que

$$A'A_1 \cdot A'A_0 = (A'A_1)^2.$$

Proof. Les centres I et I_a des cercles (Γ) et (Γ_a) appartiennent à la bissectrice AI . Par conséquent, les deux tangentes intérieures communes à ces deux cercles sont symétriques par rapport à AI .

Comme BC est l'une de ces tangentes, l'autre, notée (D') , vérifie

$$(D', AI) = (AI, BC) \pmod{\pi}.$$

Mais la question précédente a donné

$$(D, AI) = (AI, BC) \pmod{\pi}.$$

Donc

$$(D, AI) = (D', AI) \pmod{\pi},$$

ce qui entraîne

$$(D) \parallel (D').$$

- (9) Conclure

Proof. Posons $A_0 = (BC) \cap (D')$, où (D') est la seconde tangente intérieure commune à (Γ) et (Γ_a) .

D'après la question précédente, on a

$$A'A_1 \cdot A'A_0 = (A'A_1)^2.$$

Or, d'après la question 5,

$$A'A_1 = A'A'',$$

où $A'A''^2$ est la puissance de l'inversion i . Ainsi

$$A'A_1 \cdot A'A_0 = A'A''^2.$$

Par définition de l'inversion, cela montre que $A_0 = i(A_1)$.

Comme $A_1 \in (\Gamma)$ et que (Γ) est invariant par i , on en déduit que

$$A_0 \in (\Gamma).$$

Or $A_0 \in (D')$ par définition. Donc la droite (D') coupe (Γ) en A_0 .

D'autre part, on a montré que

$$(D) \parallel (D').$$

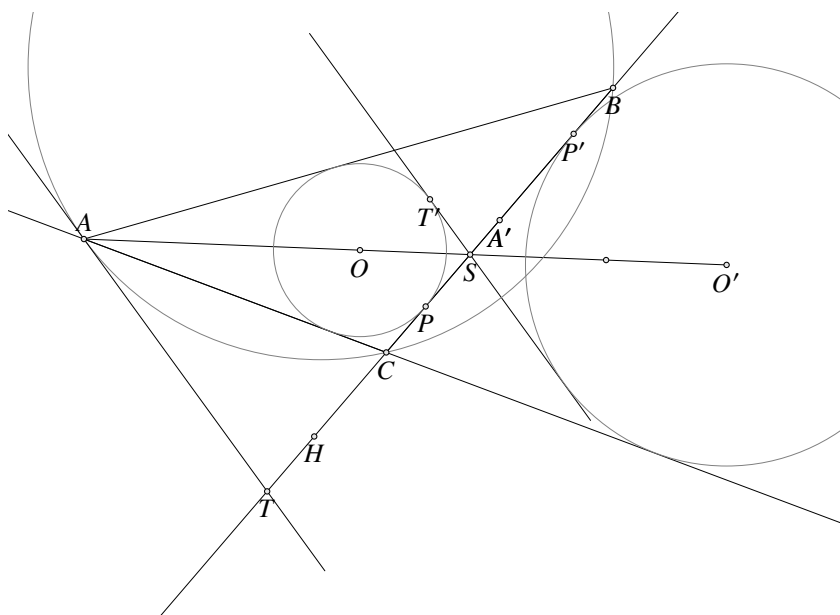
Comme $(D) = i(\Omega)$, l'image du cercle d'Euler par l'inversion est une droite parallèle à (D') ; cette droite est donc tangente à (Γ) , puisque (D') est l'une des tangentes communes à (Γ) et (Γ_a) . Par conséquent, le cercle Ω est tangent à (Γ) .

Ainsi, le cercle d'Euler est tangent au cercle inscrit.

En remplaçant (Γ_a) par chacun des trois cercles exinscrits, on obtient de la même manière que le cercle d'Euler est tangent à chacun d'eux.

Le cercle d'Euler est donc tangent au cercle inscrit et aux trois cercles exinscrits. ■

26.3. Feuerbach_Hioux

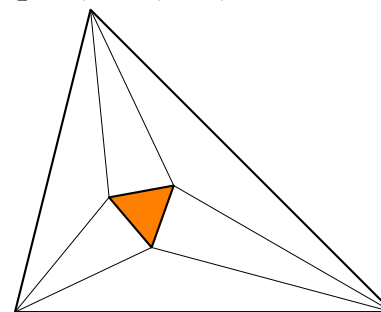


27. Morley Triangle

27.1. Morley triangle

File morley.lua:

```
init_elements()
z.A = point(0, 0)
z.B = point(5, 0)
z.C = point(1, 4)
T.ABC = triangle(z.A, z.B, z.C)
z.a1,z.a2 = tkz.trisector(z.A, z.B, z.C)
z.b1,z.b2 = tkz.trisector(z.B, z.C, z.A)
z.c1,z.c2 = tkz.trisector(z.C, z.A, z.B)
z.D = intersection_ll_(z.A, z.a1, z.B, z.b2)
z.E = intersection_ll_(z.B, z.b1, z.C, z.c2)
z.F = intersection_ll_(z.C, z.c1, z.A, z.a2)
```



28. Thébault Problems

28.1. Thébault's problem III

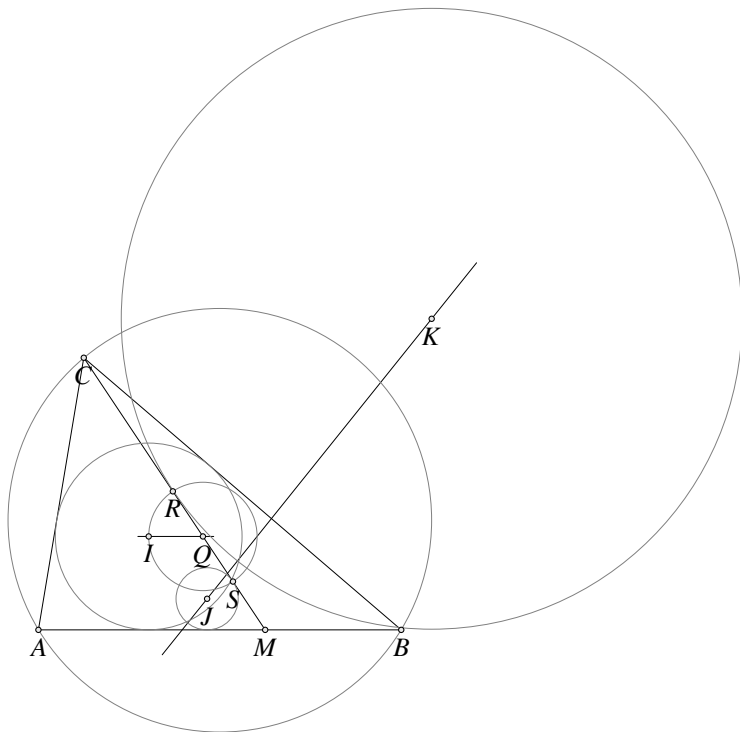
Theorem 28.1 (Thébault's problem III). *Given any triangle ABC, and any point M on BC, construct the incircle and circumcircle of the triangle. Then construct two additional circles, each tangent to AM, BC, and to the circumcircle. Then their centers and the center of the incircle are collinear.[wikipedia]*

```
init_elements()
z.A = point(0, 0)
z.B = point(8, 0)
z.C = point(1, 6)
z.M = point(5, 0)
L.CM = line(z.C, z.M)
T.ABC = triangle(z.A, z.B, z.C)
C.circ = T.ABC:circum_circle()
z.O = C.circ.center
C.ins = T.ABC:in_circle()
z.I = C.ins.center
z.T = C.ins.through
L.ll = T.ABC.ab:ll_from(z.I)
```

```

z.Q = intersection(L.11, L.CM)
C.QI = circle(z.Q, z.I)
z.R, z.S = intersection(C.QI, L.CM)
L.BMC = tkz.bisector(z.M, z.B, z.C)
z.x = L.BMC.pb
L.CMA = tkz.bisector(z.M, z.C, z.A)
z.y = L.CMA.pb
L.pS = L.CM:ortho_from(z.S)
L.pR = L.CM:ortho_from(z.R)
z.J = intersection(L.pS, L.CMA)
z.K = intersection(L.pR, L.BMC)

```

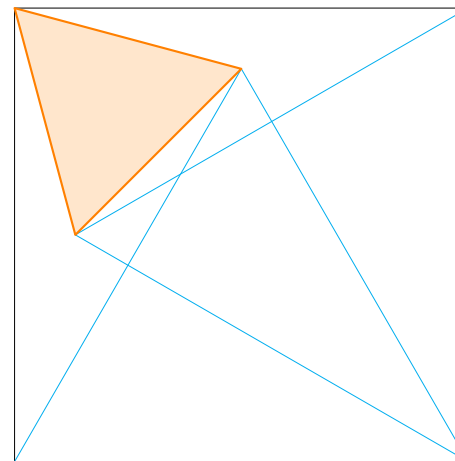


28.2. Thébault's problem II

Theorem 28.2 (Thébault's problem II). *Given a square, construct equilateral triangles on two adjacent edges, either both inside or both outside the square. Then the triangle formed by joining the vertex of the square distant from both triangles and the vertices of the triangles distant from the square is equilateral.*[wikipedia]

```
init_elements()
z.A = point(0, 0)
z.B = point(8, 0)
L.AB = line(z.A, z.B)
S.ABCD = L.AB:square()
```

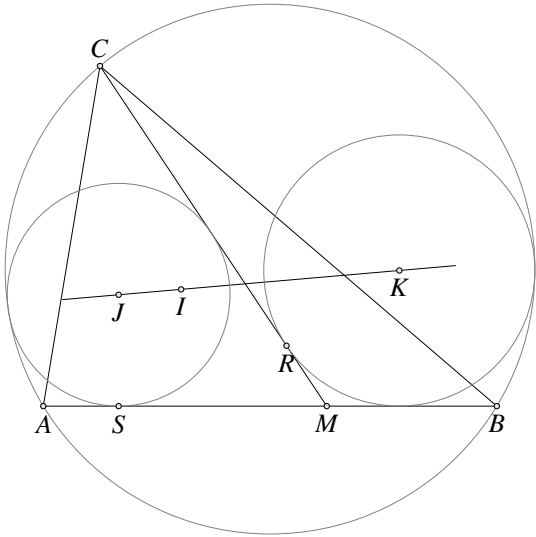
```
z.C = S.ABCD.pc
z.D = S.ABCD.pd
z.E = S.ABCD.ab:equilateral().pc
z.F = S.ABCD.bc:equilateral().pc
```



28.3. Thébault's problem III with a method

The `|c_c|` or `|thebault|` method selects a vertex of a triangle to construct a circle tangent to the two adjacent sides and to a given circle passing through the opposite vertices.

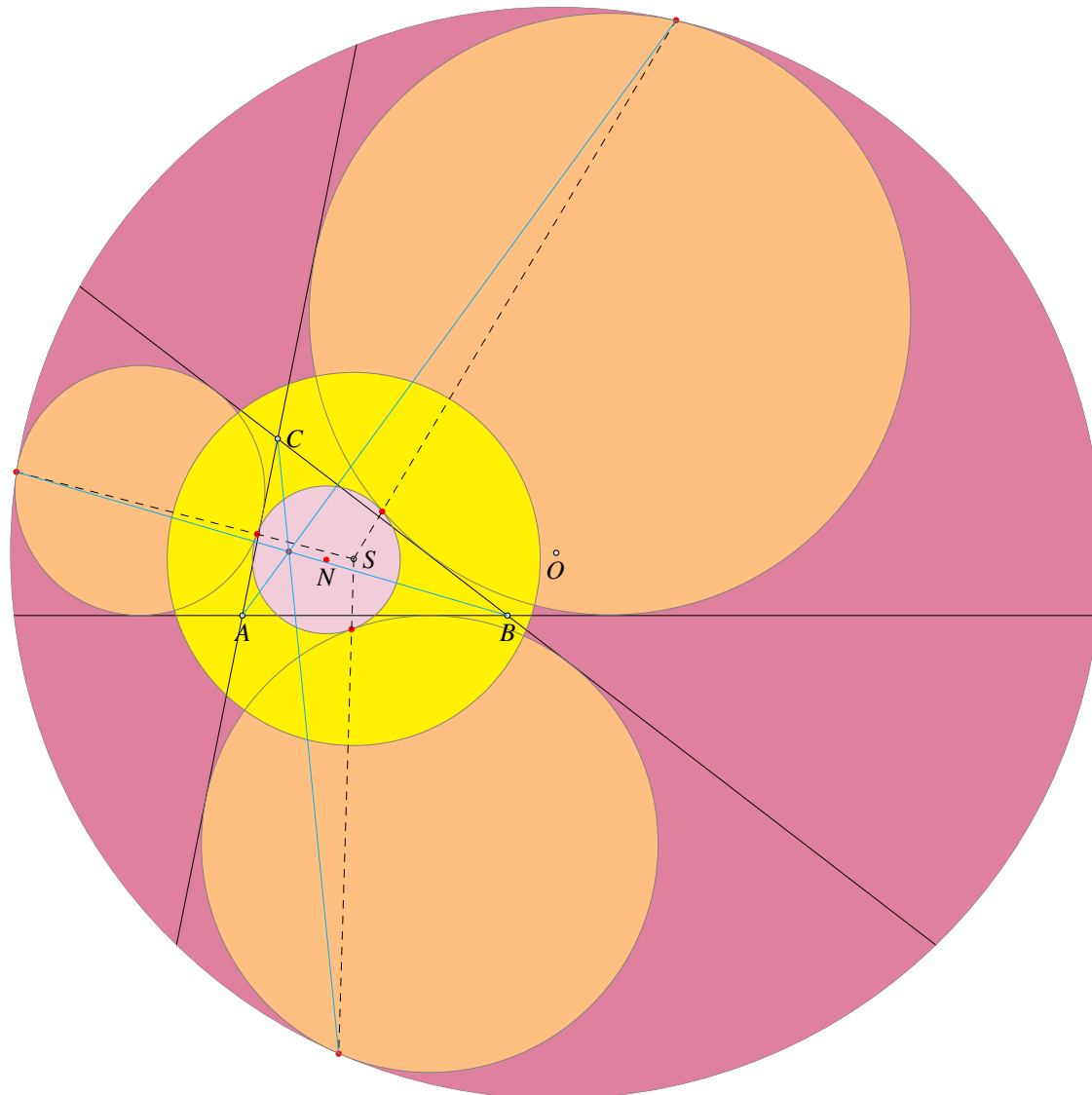
```
init_elements()
z.A = point(0, 0)
z.B = point(8, 0)
z.C = point(1, 6)
z.M = point(5, 0)
T.ABC = triangle(z.A, z.B, z.C)
C.circ = T.ABC:circum_circle()
z.O = C.circ.center
C.ins = T.ABC:in_circle()
z.I = C.ins.center
z.T = C.ins.through
T.MCA = triangle(z.M, z.C, z.A)
T.MBC = triangle(z.M, z.B, z.C)
z.J, z.S = T.MCA:thebault(z.M, C.circ):get()
z.K, z.R = T.MBC:thebault(z.M, C.circ):get()
```



29. The Apollonius Circle of a Triangle

29.1. Apollonius circle v1 with inversion

File `apo_inv.lua`:

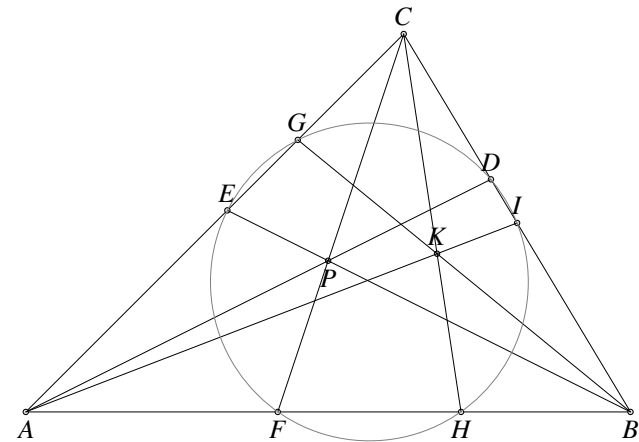


30. Classical Theorems

30.1. Reuschle's theorem

Theorem 30.1 (Reuschle's theorem). *In elementary geometry, Reuschle's theorem describes a property of the cevians of a triangle intersecting in a common point and is named after the German mathematician Karl Gustav Reuschle (1812–1875). It is also known as Terquem's theorem after the French mathematician Olry Terquem (1782–1862), who published it in 1842.*

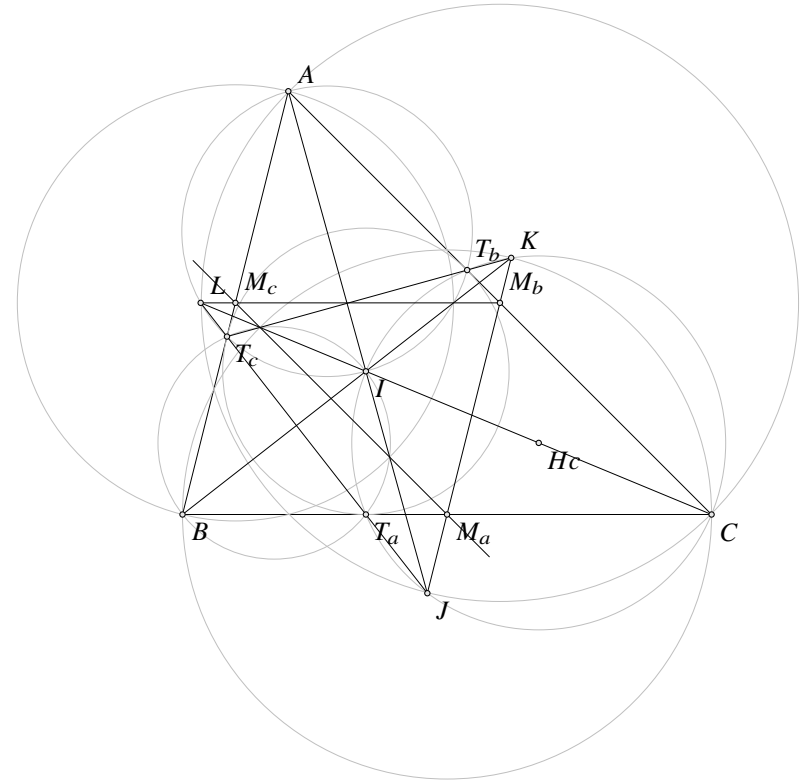
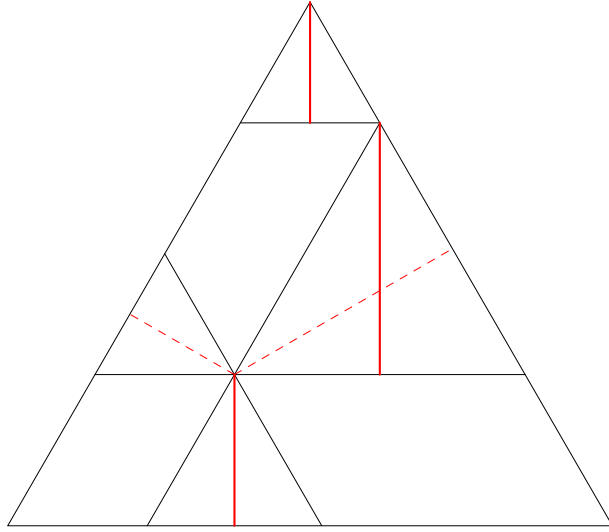
```
init_elements()
z.A = point(0, 0)
z.B = point(8, 0)
z.C = point(5, 5)
z.P = point(4, 2)
T.ABC = triangle(z.A, z.B, z.C)
T.cev = T.ABC:cevian(z.P)
z.D, z.E, z.F = T.cev:get()
C.cev = T.ABC:cevian_circle(z.P)
z.O = C.cev.center
z.T = C.cev.through
z.G = intersection(C.cev, T.ABC.ca)
_, z.H = intersection(C.cev, T.ABC.ab)
z.I = intersection(C.cev, T.ABC.bc)
L.AI = line(z.A, z.I)
L.BG = line(z.B, z.G)
z.K = intersection(L.AI, L.BG)
```



30.2. Viviani's Theorem

Theorem 30.2. *Viviani's Theorem Viviani's theorem, named after Vincenzo Viviani, states that the sum of the shortest distances from any interior point to the sides of an equilateral triangle equals the length of the triangle's altitude. [Wikipedia].*

Here's the visual demonstration I gave at the CAPES oral exam in 1989 to become a teacher.



Proof. $\widehat{GDC} = 180^\circ - \widehat{GDB} = 180^\circ - \widehat{GFB} = \widehat{AFE} = \widehat{AEF} = \widehat{GEC}$
 $\widehat{GDC} = \widehat{GEC}$

The points I, E, G, C and D are concyclic. The quadrilateral $IGCD$ is inscriptible then $\widehat{BGC} = \widehat{IGC} = \widehat{CDI} = 90^\circ$

M is the midpoint of the hypotenuse $[BC]$. $\widehat{GMC} = 2\widehat{GBC} = \widehat{ABC}$. Thus the lines (AB) and (GM) are parallel. Then G, N and M are aligned. ■

30.3. Iran Lemma

Let ABC be a triangle. Let I be the incenter, M_a, M_b, M_c be the midpoints of $[BC]$, $[CA]$, $[AB]$ and let T_a, T_b, T_c be the points of tangency of the incircle with $[BC]$, $[CA]$, $[AB]$. Then (AI) , M_aM_b , T_aT_c , the circle $C(G, A)$ and the circle $C(M_b, A)$ concur.

File `iran.lua`:

30.4. Reim's theorem 1

Theorem 30.3 (Reim's theorem). *Given two circles of center A and B intersecting at points C and D , and E and F any two points chosen on one of the circles, the lines (EC) and (FD) intersect the second circle at points H and G . We show that the lines (EF) and (HG) are parallel.*

Converse - Suppose we have two circles ω_1, ω_2 that intersect at two points C, D . Let the line through C intersect the two circles at E, H (ω_1, ω_2 respectively). Then suppose a line that cuts ω_1 at F and ω_2 at G in a way that $EF \parallel HG$. Then that line passes through D .

Proof. (EF) and (CD) are antiparallel with respect to the sides of an angle O . (CD) and (GH) are antiparallel with respect to the sides of an angle O . (EF) and (GH) are parallel. (Furthermore, if (PQ) and (RS) are antiparallel, then the points P, Q, R , and S are concyclic)

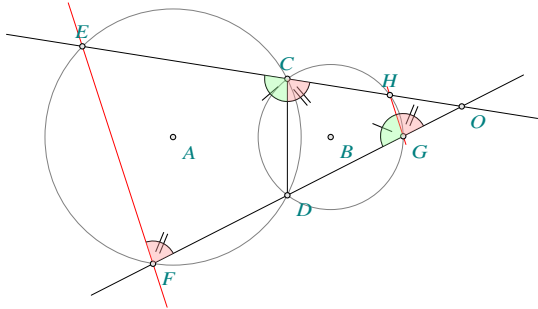
Converse - Suppose that the line (FD) intersects the circle of center B at K . Through H there is only one parallel to (EF) so G, H, K are aligned. As H and K are two points of the circle they are coincident. ■

File reim.lua

```

init_elements()
z.A = point(0, 0)
z.E = point(-2, 2)
C.AE = circle(z.A, z.E)
z.C = C.AE:point(-0.3)
z.D = C.AE:point(0.55)
z.F = C.AE:point(0.35)
L.EC = line(z.E, z.C)
z.H = L.EC:point(1.5)
T.CDH = triangle(z.C, z.D, z.H)
z.B = T.CDH.circumcenter
C.BD = circle(z.B, z.D)
L.FD = line(z.F, z.D)
z.G = intersection(L.FD, C.BD)
z.O = intersection(L.EC, L.FD)

```



30.5. Reim's theorem 2

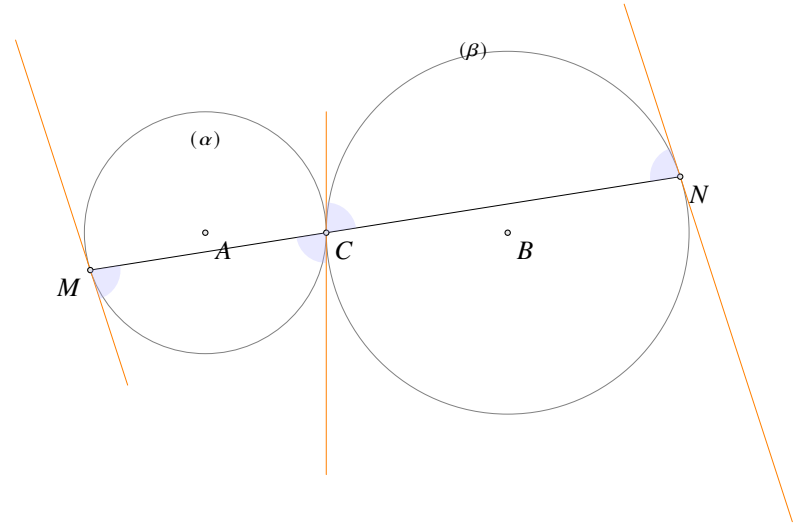
Let two circles (α) and (β) be tangent at C . Let a line passing through C intersect the circle (α) at M and intersect (β) at N . The tangents at M and N respectively to the circles (α) and (β) are parallel.

File reim2.lua

```

init_elements()
z.A = point(0, 0)
z.B = point(10, 0)
z.C = point(4, 0)
C.AC = circle(z.A, z.C)
z.c, z.cp = C.AC:tangent_at(z.C):get()
z.M = C.AC:point(0.55)
L.MC = line(z.M, z.C)
C.BC = circle(z.B, z.C)
z.N = intersection(L.MC, C.BC)
z.m, z.mp = C.AC:tangent_at(z.M):get()
z.n, z.np = C.BC:tangent_at(z.N):get()

```



Given a cyclic quadrilateral $ABQP$ and points P' and Q' on the extensions of $[PA]$ and $[FB]$, respectively. If $(PQ) \parallel (P'Q')$, then quadrilateral $ABQ'P'$ is cyclic.

(PQ) is parallel to $(P'Q')$ if and only if the points A, P', Q' and B are cocyclic.

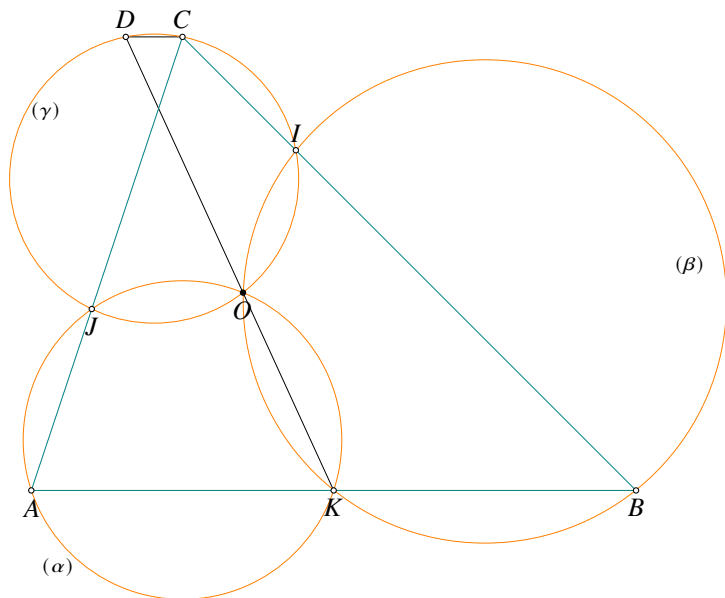
It is sufficient to show that $\widehat{ABQ'} = \widehat{AP'Q'}$.

File reim2a.lua

```

init_elements()
z.A = point(0, 0)
z.B = point(8, 0)
z.C = point(2, 6)
L.AB = line(z.A, z.B)
L.AC = line(z.A, z.C)
L.BC = line(z.B, z.C)
z.I = L.BC:point(0.75)
z.J = L.AC:point(0.4)
z.K = L.AB:point(0.5)
T.AKJ = triangle(z.A, z.K, z.J)
T.BIK = triangle(z.B, z.I, z.K)
T.CIJ = triangle(z.C, z.I, z.J)
z.x = T.AKJ.circumcenter
z.y = T.BIK.circumcenter
z.z = T.CIJ.circumcenter
C.xK = circle(z.x, z.K)
C.yK = circle(z.y, z.K)
z.O, _ = intersection(C.xK, C.yK)
C.z0 = circle(z.z, z.O)
L.KO = line(z.K, z.O)
z.D = intersection(L.KO, C.z0)

```



K, O the intersection points of (α) and (β) ,
 I one of the two points of intersection of (β) and (γ) ,
 J one of the two points of intersection of (γ) and (α) ,
 A a point of (α) ,
 B the second point of intersection of (AK) with (β)
 C the second point of intersection of (BI) with (γ) .

Let D be the second point of intersection of the line (KO) with (γ) Reim's theorem applied to the circles (α) and (γ) .

The circles (γ) and (β)

Given a cyclic quadrilateral $ABFE$ and points G and H on the extensions of (EA) and (FB) , respectively. If $GH \parallel EF$, then quadrilateral $ABHG$ is cyclic.

Reim's theorem $(CD) \parallel (AB)$

30.6. Three chords

Theorem 30.4 (Three chords_Monge). *Monge (Gaspard (1746 - 1818) Three circles intersect pairwise but let's assume there is no point shared by all three of them. There are three pairs of circles. Two circles in any pair share a chord. The problem is to prove that the three chords meet at a point.*

β, γ two intersecting circles A, B the points of intersection of β and γ , C, D two points of γ , E, F two points of β , and I the point of intersection of (AB) and (CD) .

Question : C, D, E, F are cocyclic if, and only if, (EF) passes through I . Let us note: (α) the circle passing through C, D, E, F . (δ) the circle passing through B, F, D . and A', C', E' the second points of intersection of $(AB), (CD), (EF)$ with (δ) .

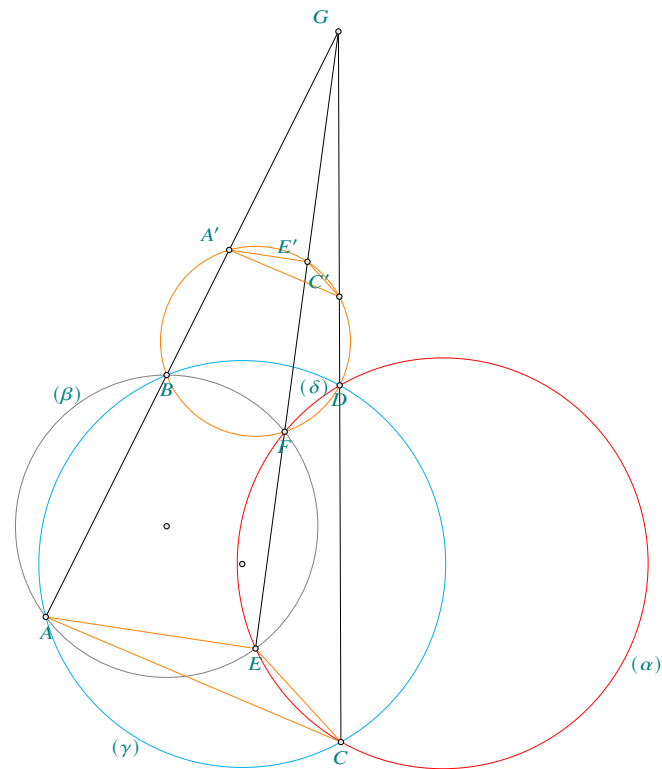
Reim's theorem $(AC) \parallel (A'C')$ $(CE) \parallel (C'E')$ $(EA) \parallel (E'A')$

According to Desargues "The weak theorem" (Cf. Appendix) applied to homothetic triangles

EAC and $E'A'C'$, (EE') passes through I . Conclusion: (EF) passes through I .

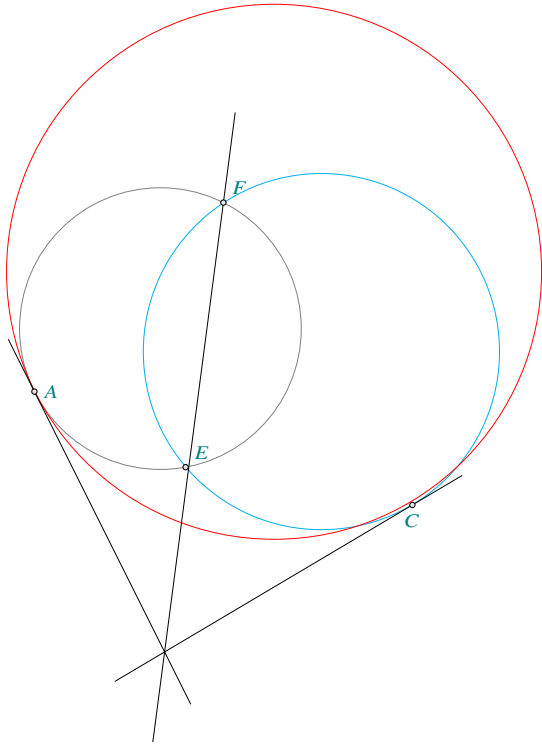
Monge's statement: when three circles intersect in pairs, the three intersecting lines are concurrent. Statement: If two circles are secant and the common line passes through the point of

intersection of a secant of one and a secant of the other, then the four points of intersection are cocyclic.



File three_chords.lua

File three_chords2.lua

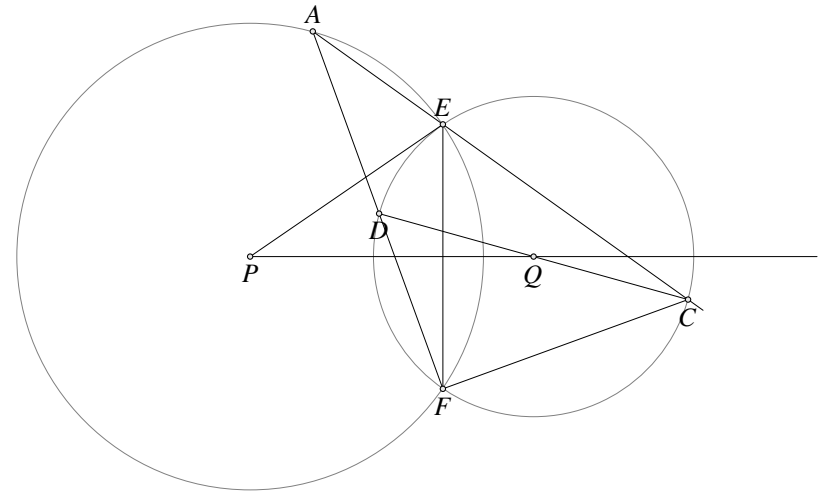


30.7. Altshiller-Court's theorem

Theorem 30.5 (Altshiller-Court's theorem_constant chord theorem). *The two lines joining the points of intersection of two orthogonal circles to a point on one of the circles meet the other circle in two diametrically opposite points. Altshiller p 176. Nathan Altshiller Court described the constant chord theorem 1925 in the article sur deux cercles secants for the Belgian math journal Mathesis.[Wikipedia]*

File altshiller.lua

```
init_elements()
z.P = point(0, 0)
z.Q = point(5, 0)
z.I = point(3, 2)
C.QI = circle(z.Q, z.I)
C.PE = C.QI:orthogonal_from(z.P)
z.E = C.PE.through
C.QE = circle(z.Q, z.E)
_, z.F = intersection(C.PE, C.QE)
z.A = C.PE:point(1 / 9)
L.AE = line(z.A, z.E)
_, z.C = intersection(L.AE, C.QE)
L.AF = line(z.A, z.F)
L.CQ = line(z.C, z.Q)
z.D = intersection(L.AF, L.CQ)
```



Proof. Let the lines (EA) , (FA) joining the points of intersection E, F of the two orthogonal circles (P) , (Q) to the point A of (P) met (Q) again in C, D . We have:

$$\widehat{EAF} = \frac{1}{2}\widehat{EPF} = \widehat{EPQ} \text{ and } \widehat{ECF} = \frac{1}{2}\widehat{EQF} = \widehat{EQP}$$

Hence $\widehat{EAF} + \widehat{ECF} = \widehat{EPQ} + \widehat{EQP} = 90^\circ$ and therefore the triangle CAF is right-angled at F . Thus CD subtends a right angle at F , which proves the proposition. ■

30.8. D'Alembert's Theorem

Theorem 30.6 (D'Alembert-Monge). *In geometry, Monge's theorem, named after Gaspard Monge, states that for any three circles in a plane, none of which is completely inside one of the others, the intersection points of each of the three pairs of external tangent lines are collinear.[Wikipedia]. If three circles A, B, and C are taken in pairs, the external similarity points of the three pairs lie on a straight line (I, J, K are aligned). Similarly, the external similarity point of one pair and the two internal similarity points of the other two pairs lie upon a straight line, forming a similarity axis of the three circles (Two inner centers are aligned with an outer center: I', K', J or I', J', K or I, J', K').*

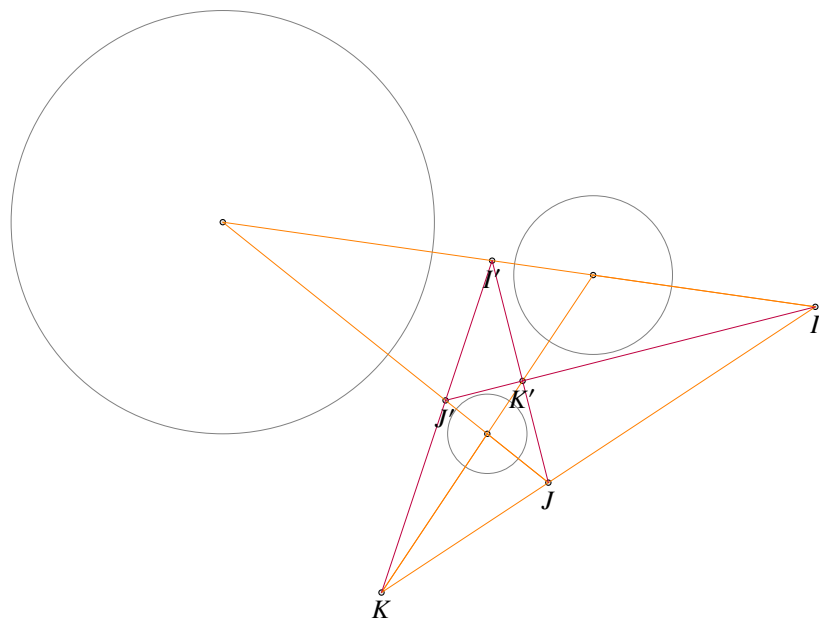
File alembert.lua:

```
init_elements()
z.A = point(0, 0)
z.a = point(4, 0)
z.B = point(7, -1)
z.b = point(5.5, -1)
z.C = point(5, -4)
z.c = point(4.25, -4)
C.Aa = circle(z.A, z.a)
C.Bb = circle(z.B, z.b)
C.Cc = circle(z.C, z.c)
z.I = C.Aa:external_similitude(C.Bb)
z.J = C.Aa:external_similitude(C.Cc)
z.K = C.Cc:external_similitude(C.Bb)
```

```

z.Ip = C.Aa:internal_similitude(C.Bb)
z.Jp = C.Aa:internal_similitude(C.Cc)
z.Kp = C.Cc:internal_similitude(C.Bb)

```

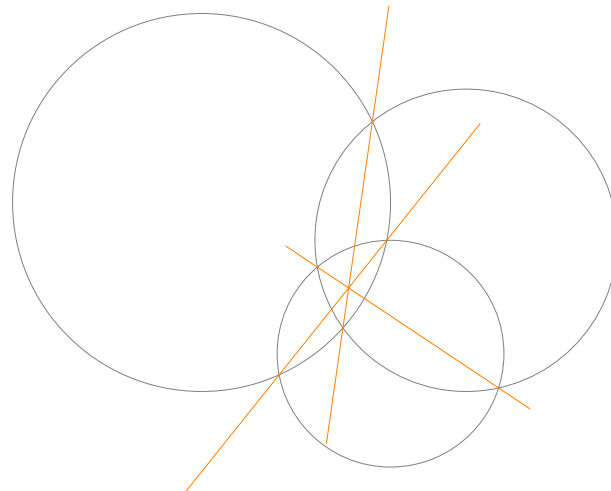


File alembert2.lua:

```

init_elements()
z.A = point(0, 0)
z.a = point(5, 0)
z.B = point(7, -1)
z.b = point(3, -1)
z.C = point(5, -4)
z.c = point(2, -4)
C.Aa = circle(z.A, z.a)
C.Bb = circle(z.B, z.b)
C.Cc = circle(z.C, z.c)
z.i, z.j = C.Aa:radical_axis(C.Bb):get()
z.k, z.l = C.Aa:radical_axis(C.Cc):get()
z.m, z.n = C.Bb:radical_axis(C.Cc):get()

```



30.9. Archimedes' Book of Lemmas: Proposition 1

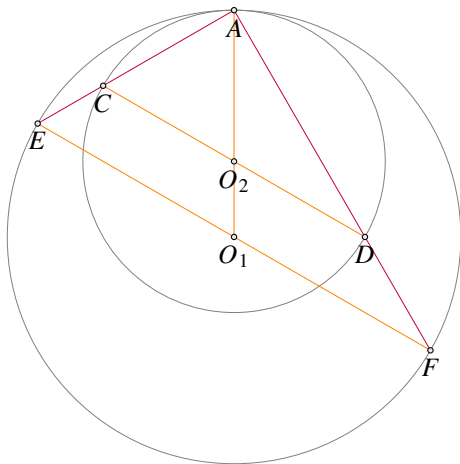
Theorem 30.7 (Archimedes). *If two circles touch at A, and if $[CD]$, $[EF]$ be parallel diameters in them, the points A, C, and E are aligned.*

File archimedes.lua:

```

init_elements()
z.O_1 = point(0, 0)
z.O_2 = point(0, 1)
z.A = point(0, 3)
z.F = point:polar(3, -math.pi / 6)
L.F01 = line(z.F, z.O_1)
C.O1A = circle(z.O_1, z.A)
z.E = intersection(L.F01, C.O1A)
T.ABC = triangle(z.F, z.E, z.O_2)
z.x = T.ABC:parallelogram()
L.F02 = line(z.x, z.O_2)
C.O2A = circle(z.O_2, z.A)
z.C, z.D = intersection(L.F02, C.O2A)

```



Proof. $(CD) \parallel (EF)$

(AO_0) is secant to these two lines so $\widehat{AO_2C} = \widehat{AO_1E}$.

Since the triangles AO_1C and AO_0E are isosceles the angles at the base are equal $\widehat{AC}O_2 = \widehat{AE}O_1 = \widehat{CA}O_2 = \widehat{EA}O_1$. Thus A, C and E are aligned. ■

31. Apollonius' problem

Overview

The classical Apollonius' problem asks for one or several circles tangent to three given geometric objects, each of which may be a circle (C), a line (L) or a point (P). Each case is denoted by a three-letter code such as CCP, LLL, or PLC, according to the type of the three given objects.

Historically, there are ten distinct configurations:

PPP, LPP, CPP, LLP, LLL, CLP, CLL, CCP, CCL, CCC.

Viète solved all ten cases using only compass and straightedge constructions, building complex solutions from simpler ones.

Note (work in progress): This section is under active development. Some *degenerate* or *very special* configurations may not be fully handled yet.

Unified API (paths everywhere)

Each case is implemented as a method attached to the class of the *first* object (e.g. `point:PPP`, `line:LLL`, `circle:CLL`). All methods are being unified to return *paths* so that drawing and node extraction are consistent across cases.

Syntax: `PA.center`, `PA.through`, `n = <obj>:XYZ(args)`

Returned values:

- `PA.center` : a path collecting the centers of the solution circles;
- `PA.through` : a path collecting the corresponding through-points;
- `n` : the number of valid solutions (can be 0).

Remarks:

- All results can be directly drawn using `\tkzDrawCirclesFromPaths(PA.center, PA.through)`.
- Nodes can be exported to TikZ with `tkz.nodes_from_paths(PA.center, PA.through)`.
- Empty paths correspond to degenerate configurations.
- Some older methods still return `paths`, `n`; they will be progressively updated to the unified format.

Using `tkz.nodes_from_paths` instead of `\tkzDrawCirclesFromPaths`

If you wish to draw *all* the solution circles at once, the macro

`\tkzDrawCirclesFromPaths(PA.ce, PA.th)` is the most convenient approach.

However, if you need to highlight or draw *only one specific circle*, it is often preferable to use the Lua helper `tkz.nodes_from_paths`. In this case, you may keep the solutions in local variables to avoid polluting the global table:

```
local pc, pt, n = C.OG:CPP(z.A, z.B)
tkz.nodes_from_paths(pc, pt)
% creates z.w1, z.t1, z.w2, z.t2,...
```

By default, this generates nodes $(w1, t1)$, $(w2, t2)$, ... in the table `z`, which can then be used to draw individual circles or to label / style them differently.

Combining both approaches: You can also combine the two techniques:

first use `\tkzDrawCirclesFromPaths` to display all solution circles, and then use `tkz.nodes_from_paths` to extract the corresponding nodes and emphasize only some of them

(change colour, add labels, draw radii, etc.). This is particularly useful in demonstrations or pedagogical figures.

Example idea:

- Compute the solutions:
`PA.center, PA.through, n = C.OC:CPP(z.A, z.B).`
- Draw all circles with `\tkzDrawCirclesFromPaths`.
- Call `tkz.nodes_from_paths(PA.center, PA.through)`.
- Then, for instance: `\draw[red,thick] (z.w1) circle[radius];` to highlight only the first one.

Thus, the macro gives a quick visual result, while `tkz.nodes_from_paths` offers fine control over individual solutions.

Global PA.xxx vs Local Variables:

When using contact methods such as:

```
PA.center, PA.through, n = C.OC:CPP(z.A, z.B)
```

the two paths `PA.center` and `PA.through` are stored as *global* variables in the `z/PA` namespace.

This has two important advantages:

- They can be used directly in TikZ with macros such as:
`\tkzDrawCirclesFromPaths(PA.center, PA.through)`.
- They are automatically cleared when calling `init_elements()`, which avoids name conflicts between different figures or compilations.

Since TikZ macros work purely on the TeX side, the paths must exist globally to be accessible outside the `\directlua` block.

Using Local Paths with `tkz.nodes_from_paths`:

If you do not want to pollute the global namespace, you may store the results locally:

```
local pc, pt, n = C.OC:CPP(z.A, z.B)
```

```
tkz.nodes_from_paths(pc, pt) -- creates z.w1, z.t1,...
```

Here, `pc` and `pt` remain Lua-local, but the function `tkz.nodes_from_paths` exports the corresponding nodes (`w1, t1, w2, t2, ...`) into the global table `z` so they can be used in TikZ.

Which One to Use?

- Use `PA.center, PA.through` (global) if you plan to call:
`\tkzDrawCirclesFromPaths(PA.center, PA.through)`

because this macro expects global paths accessible from TeX.

- Use `tkz.nodes_from_paths` if you want to select or highlight only some of the solution circles. It also works with either global or local paths.
- A combined approach is often the most flexible: draw all circles with the macro, then export nodes to highlight specific ones.

Example Workflow:

- (1) Compute solutions globally:
`PA.center, PA.through, n = C.OC:CPP(z.A, z.B)`
- (2) Draw all circles:
`\tkzDrawCirclesFromPaths(PA.center, PA.through)`
- (3) Export the same centers and points to nodes:
`\directlua{ tkz.nodes_from_paths(PA.center, PA.through) }`

- (4) Now highlight one circle, for instance:

```
\draw[red,thick] (z.w1) -- (z.t1);
```

In summary, `PA.xxx` must be *global* for TikZ macros, while `tkz.nodes_from_paths` lets you create node names (`w1, t1, ...`) from either global or local paths depending on how much control you need.

31.1. Class point: method PPP(p, p)

Purpose:

The method PPP constructs the *circumcircle* passing through three distinct points: the current point (`self`) and the two given points `a` and `b`.

If the three points are collinear or not distinct, no circle can be defined and the method returns empty paths with a counter equal to 0.

Syntax: `PA.center, PA.through, n = z.A:PPP(z.B, z.C)`

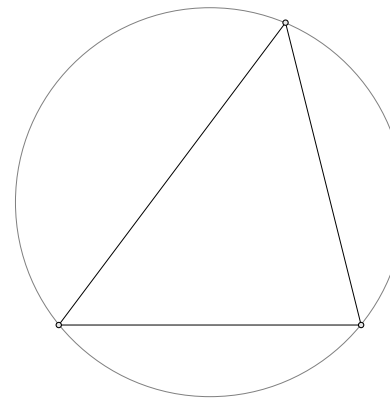
Returns:

- `PA.center`: a path containing the center(s) of the circumcircle(s);
- `PA.through`: a path containing the corresponding point(s) on each circle;
- `n`: the number of valid circles (1 if a circumcircle exists, 0 otherwise).

Example usage:

File `apo_PPP.lua`:

```
init_elements()
z.A = point(0, 0)
z.B = point(4, 0)
z.C = point(3, 4)
PA.center, PA.through, _ = z.A:PPP(z.B, z.C)
tkz.nodes_from_paths(PA.center, PA.through)
-- w and t by default
```



Remarks: This is the simplest case of the Apollonius family. If `n = 0`, the three points are collinear and no circle is drawn.

31.2. Class line: method LPP(p, p)

Purpose: Given a running line L (the current object `self`) and two points a and b lying on the *same side* of L , this method constructs the circle(s) tangent to L and passing through both a and b .

Syntax: `local PA.center, PA.through, n = L:LPP(a, b)`

Parameters:

- a, b : two distinct points.
- *Precondition:* a and b are on the same side of the running line `self`.

The method is designed to handle a variety of geometric configurations, including the following special cases:

- The segment $[MN]$ is perpendicular to the line (AB) ;
- The segment $[MN]$ is parallel to the line (AB) ;
- The points M and N lie on opposite sides of the line (AB) ;
- One of the points (either M or N) lies on the line (AB) .

This construction is useful in problems involving constrained circle placements or Apollonius-type configurations.

Returns:

- `PA.center` and `PA.through` : two paths containing respectively the centers and the points of tangency of the solution circles.
- n : the number of existing solutions (0, 1 or 2).

Remarks:

- If $n = 0$, both paths are empty and no circle can be drawn.
- In degenerate cases with a single solution, the same circle is recorded once ($n = 1$).
- No `tex.error` is raised; the method always returns valid paths.
- The results can be transferred and drawn using the macros:
`\tkzNodesFromPaths` and `\tkzDrawCirclesFromPaths`.

Existence & number of solutions

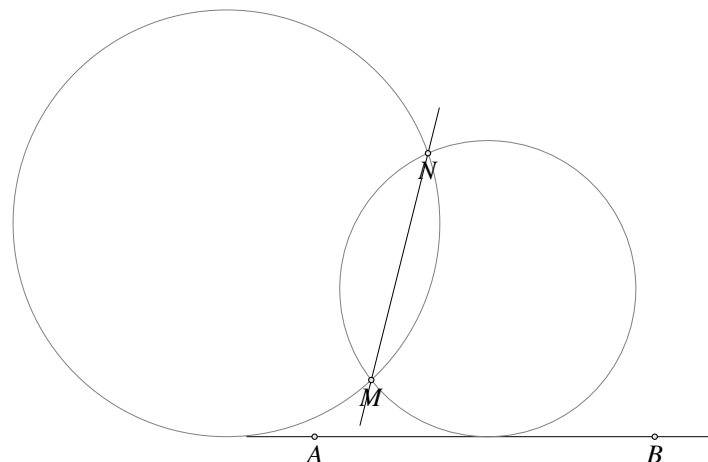
- No solution (0). If the intersection point $i = L \cap (ab)$ exists and lies on the segment $[a, b]$, then no circle tangent to L can pass through a and b .
- Single solution (1).
 - If exactly one of the points a or b belongs to L (point of tangency), the unique solution has its center at the intersection of the mediator of $[a, b]$ with the perpendicular to L at that point.
 - If $L \parallel (ab)$, the unique solution has its center at the circumcenter of the triangle (a, b, p) where p is the orthogonal projection of $\text{mid}(a, b)$ on L .
- Two solutions (2).
 - If $L \perp (ab)$ and $i = L \cap (ab)$ is defined (and not on $[a, b]$), two symmetric solutions exist.
 - In the general oblique case (neither parallel nor orthogonal), two solutions exist.

Example usage:

File `apo_LPP.lua`:

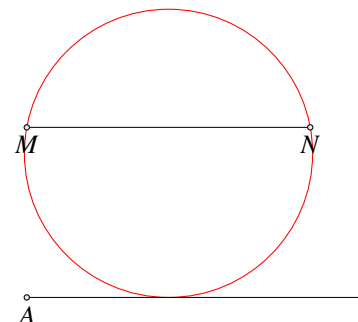
```
init_elements()
z.A = point(0, 0)
z.B = point(6, 0)
```

```
z.M = point(1, 1)
z.N = point(2, 5)
L.AB = line(z.A, z.B)
PA.center,
PA.through, n = L.AB:LPP(z.M, z.N)
tkz.nodes_from_paths(PA.center, PA.through, "O", "T")
```



Let's look at the case where the line (MN) is parallel to the initial line.
File `apo_LPP2.lua`:

```
init_elements()
z.A = point(0, 0)
z.B = point(6, 0)
z.M = point(0, 3)
z.N = point(5, 3)
L.AB = line(z.A, z.B)
PA.center, PA.through, n = L.AB:LPP(z.M, z.N)
```



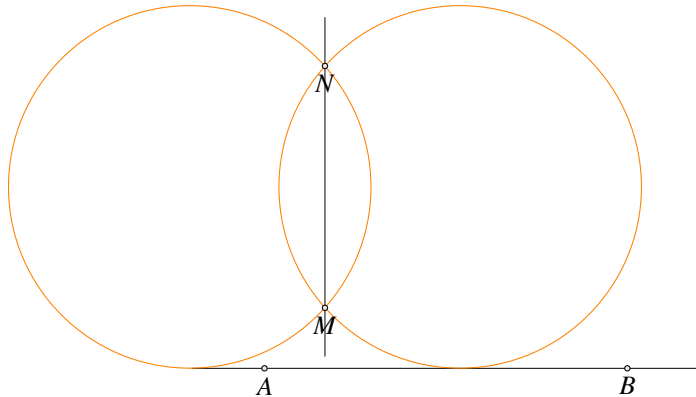
Where the line is perpendicular to the initial line.

File `apo_LPP3.lua`:

```

init_elements()
z.A = point(0, 0)
z.B = point(6, 0)
z.M = point(1, 1)
z.N = point(1, 5)
L.AB = line(z.A, z.B)
PA.center,
PA.through, n = L.AB:LPP(z.M, z.N)

```



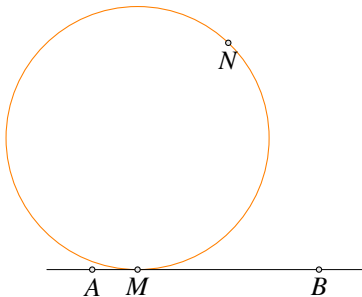
The last special case is when one of the points is on the initial line. In this case, there's only one solution.

File apo_LPP4.lua:

```

init_elements()
z.A = point(0, 0)
z.B = point(5, 0)
z.M = point(1, 0)
z.N = point(3, 5)
L.AB = line(z.A, z.B)
PA.center,
PA.through, n = L.AB:LPP(z.M, z.N)
PA.center, PA.through,
n = L.AB:LPP(z.M, z.N)

```



31.3. Class line: method LLP(L, p)

Purpose: Given the running line D (**self**), another line L , and a point p , this method constructs the circle(s) passing through p and tangent to *both* lines D and L .

Syntax: PA.center, PA.through, n = L.AB:LLP(L, p)

Let us consider two straight lines, (AB) and (CD) , and a point P not lying on either line. The problem is to determine whether there exists a circle passing through P and tangent to both lines.

Parameters:

- L : a line distinct from the running line **self** (they may be parallel or intersecting).
- p : a point in the plane.

Returns:

- **PA.center, PA.through**: two paths containing the centers and tangent points of the solution circles.
- **n**: the number of solutions (0, 1, or 2).
- If no solution exists, **n** = 0 and both paths are empty.

Remarks: This method generalizes Apollonius' problem in the case of two lines and one point. It handles both parallel and intersecting lines, as well as special configurations where the point lies on a bisector or one of the lines.

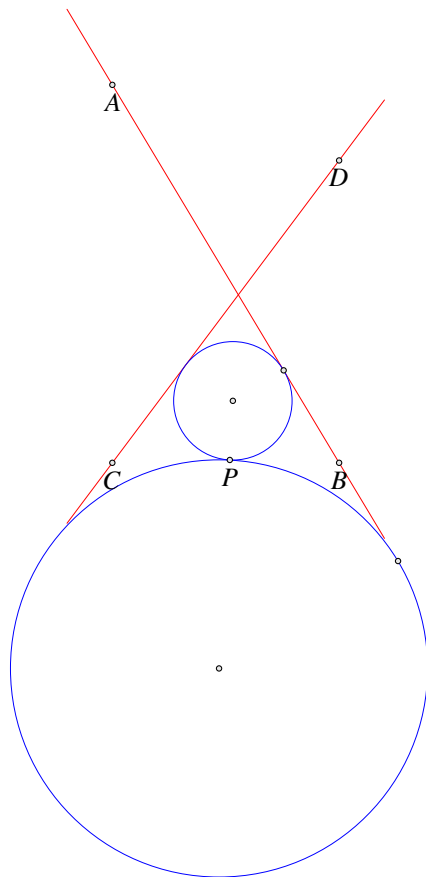
General case

File apo_LLP.lua:

```

init_elements()
z.A = point(-1, 2)
z.B = point(2, -3)
z.C = point(-1, -3)
z.D = point(2, 1)
L.AB = line(z.A, z.B)
L.CD = line(z.C, z.D)
z.S = intersection(L.AB, L.CD)
L.SI = tkz.bisector(z.S, z.A, z.D)
z.P = L.SI:point(-1)
local centers,
throughs, n = L.AB:LLP(L.CD, z.P)
tkz.nodes_from_paths(centers, throughs)

```



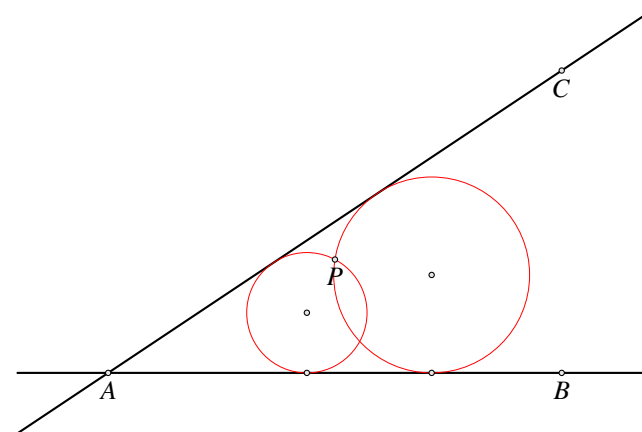
Point inside the angle

File apo_LL2.lua:

```

init_elements()
z.A = point(0, 0)
z.B = point(6, 0)
L.AB = line(z.A, z.B)
z.C = point(6, 4)
L.AC = line(z.A, z.C)
z.P = point(3, 1.5)
local centers,
throughs, n = L.AB:LLP(L.AC, z.P)
tkz.nodes_from_paths(centers, throughs)

```



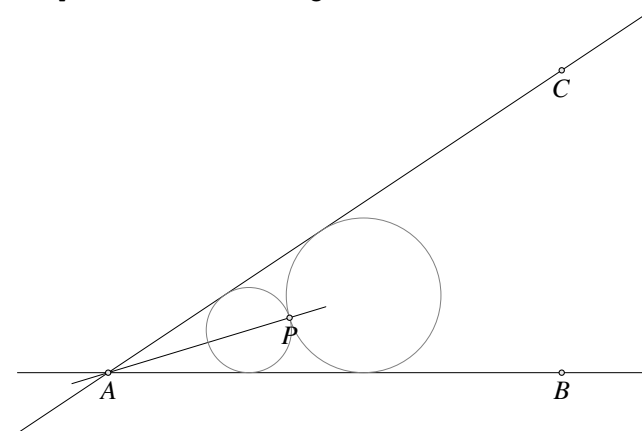
Point on a bisector

File apo_LL3.lua:

```

init_elements()
z.A = point(0, 0)
z.B = point(6, 0)
L.AB = line(z.A, z.B)
z.C = point(6, 4)
L.AC = line(z.A, z.C)
L.bi = tkz.bisector(z.A, z.B, z.C)
z.P = L.bi:point(0.4)
local centers,
throughs, n = L.AB:LLP(L.AC, z.P)
tkz.nodes_from_paths(centers, throughs)

```



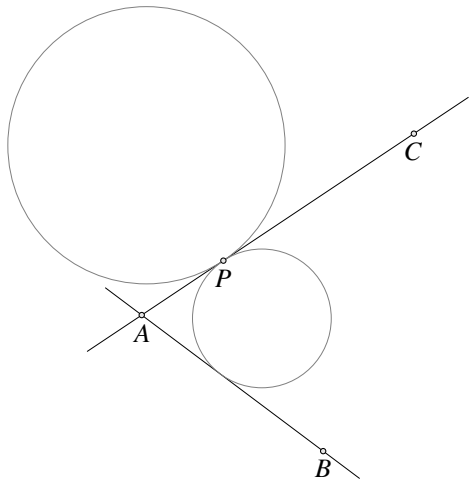
Point on one of the lines

File apo_LL4.lua:

```

init_elements()
z.A = point(0, 0)
z.B = point(4, -3)
L.AB = line(z.A, z.B)
z.C = point(6, 4)
L.AC = line(z.A, z.C)
z.P = L.AC:point(.3)
local centers,
  throughs, n = L.AB:LLP(L.AC, z.P)
tkz.nodes_from_paths(centers, throughs)

```



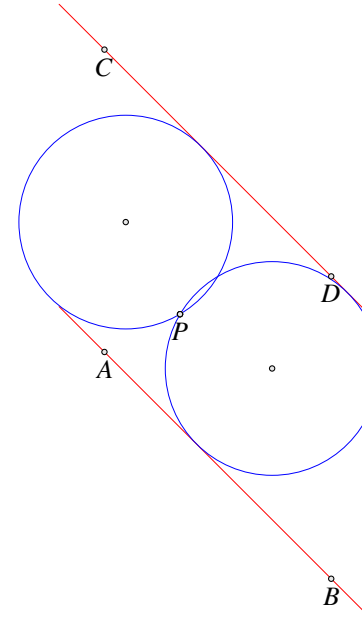
Parallel lines

File apo_LLP5.lua:

```

init_elements()
z.A = point(-2, 2)
z.B = point(1, -1)
z.C = point(-2, 6)
z.D = point(1, 3)
L.AB = line(z.A, z.B)
L.CD = line(z.C, z.D)
z.P = point(-1, 2.5)
local centers,
  throughs, n = L.AB:LLP(L.CD, z.P)
tkz.nodes_from_paths(centers, throughs)

```



31.4. Class line: method LLL(L, L)

This method completes the family of contact configurations. It handles the case of a circle tangent to three lines.

Syntax:

```

C_in = L.AB:LLL(L.BC, L.CA, "in")
C_exA = L.AB:LLL(L.BC, L.CA, "pa")
C_in, C_ea, C_eb, C_ec = L.AB:LLL(L.BC, L.CA)

```

Purpose:

The method LLL constructs the circles tangent to three lines.

- If the three lines are *non-parallel*, they define a triangle, and LLL returns the *incircle* and the three *excircles*, equivalent to `triangle:c_lll("all")`.
- If exactly two lines are parallel and the third one is transversal, two circles are tangent to the three lines; these two solutions are returned.

Returns:

- In the triangle case:
 - "all" : four circles (incircle and three excircles),
 - "in" : incircle only,
 - "pa", "pb", "pc" : the corresponding excircle.
- In the parallel case: two circles tangent to the three lines.

Errors:

If at least two pairs of lines are parallel, the configuration is degenerate.

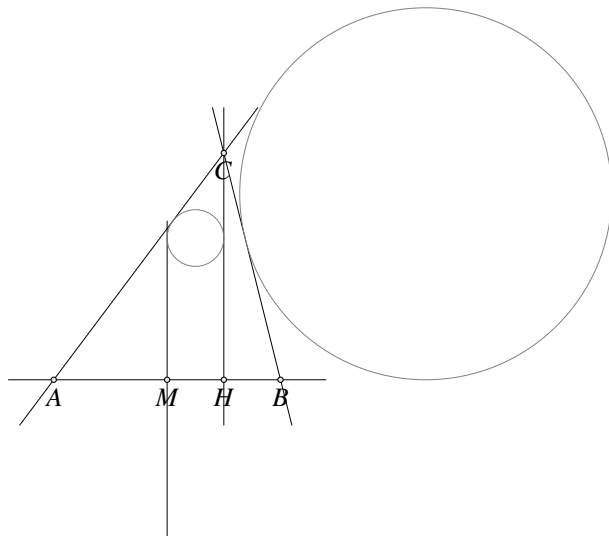
Example usage:

File apo_LLL1.lua:

```

init_elements()
z.A = point(0, 0)
z.B = point(6, 0)
z.C = point(4.5, 6)
T.ABC = triangle(z.A, z.B, z.C)
L.AB = line(z.A,z.B)
L.AC = line(z.A,z.C)
L.med = L.AB : mediator ()
z.M = L.AB.mid
z.x, z.y = L.med:get()
z.H = L.AB:projection(z.C)
L.ortho = L.AB:orthogonal_from(z.C)
PA.center, PA.through = L.AC:LLL(L.med, L.ortho)
z.w = PA.center:get(2)
z.t = PA.through:get(2)
PA.center, PA.through = L.AB:LLL(L.AC, T.ABC.bc)
z.o = PA.center:get(4)
z.h = PA.through:get(4)

```



31.5. Class: method CLL(L, L)

Alias: `c_cll`

Syntax:

`PA.center, PA.through, n = circle:CLL(L1, L2, choice, inside)`

Purpose:

Constructs the circles that are tangent to:

- the current circle (`self`),
- and the two lines `L1` and `L2`.

The method returns two `path` objects containing all the solutions found:

- `PA.center` – centers of the solution circles,

- `PA.through` – corresponding through-points (point of tangency),
- `n` – number of solutions.

Options:

- `choice` – Optional. Integer from 1 to 4 selecting one of the angular sectors formed by the two lines at their intersection. By default, `choice = "all"` which means all sectors are explored and all valid solutions are returned.
- `inside` – Optional. If set to "inside", the circle is tangent internally to the reference circle (its radius is considered negative in the auxiliary construction). If omitted, tangency is external.

Explanation:

- The intersection of `L1` and `L2` defines four sectors numbered from 1 to 4 in the direct (counterclockwise) order starting from `L1`.
- For each chosen sector, up to two circles can be tangent to the two lines and to the reference circle.
- With the default option `choice = "all"`, the method tries all four sectors and appends every valid solution to the paths.

Remarks:

- If the lines are parallel, this method does not apply.
- The user can draw the resulting circles using `\tkzDrawCirclesFromPaths(PA.center, PA.through)`.
- The value of `n` gives the total number of circles stored in the paths.

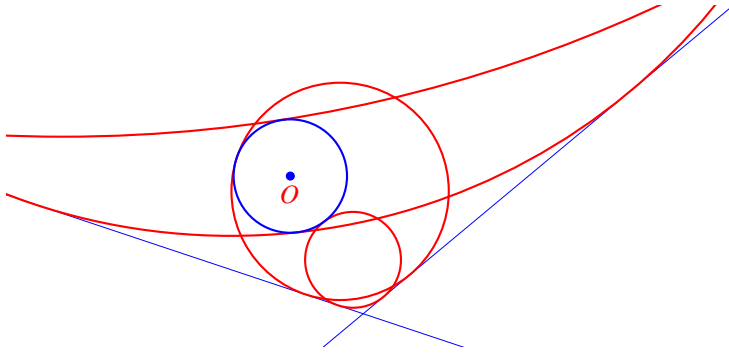
Examples usage:

File `apo_CLL1.lua`:

```

init_elements()
z.A = point(0, 0)
z.B = point(6, -2)
L.AB = line(z.A, z.B)
z.C = point(-3, -4)
z.D = point(3, 1)
L.CD = line(z.C, z.D)
z.O = z.D + point(-3,1)
z.X = z.O + point(0,1)
C.OX = circle(z.O, z.X)
PA.center, PA.through = C.OX:CLL(L.AB, L.CD,"all")
tkz.nodes_from_paths(PA.center, PA.through)

```

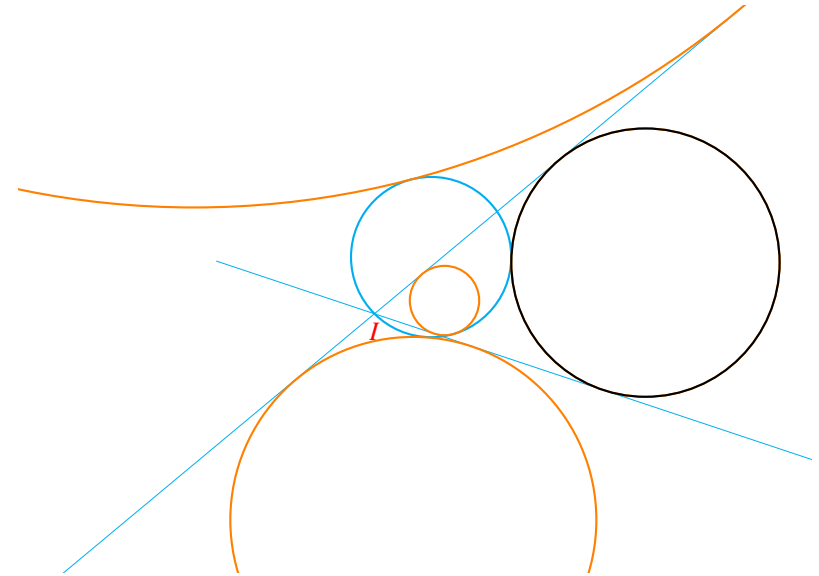


Some explanations: The given circle lies within sectors 1 and 2. In this case, we are looking for the solution circles located in these two sectors. There is a method that determines how a circle is positioned with respect to two lines: `circle:lines_position(L1, L2)`. If you do not know how many solutions exist, the following example shows how to proceed.

A special case: The intersection point of the two lines belongs to the given circle. In this example, the `CLL_all` method determines all solutions. The points that define the solution circles are stored in objects of type `path`.

File `apo_CLL2.lua`:

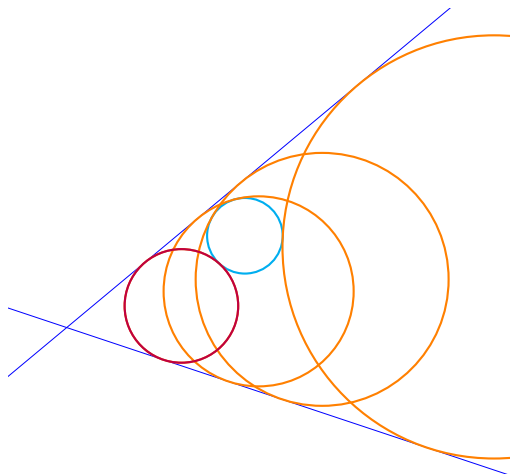
```
init_elements()
z.A = point(0, 0)
z.B = point(6, -2)
L.AB = line(z.A, z.B)
z.C = point(-3, -4)
z.D = point(3, 1)
L.CD = line(z.C, z.D)
z.I = intersection(L.AB, L.CD)
z.O = z.I + point(1,1)
z.X = z.I
C.OX = circle(z.O, z.X)
PA.center, PA.through = C.OX:CLL_all(L.AB, L.CD)
z.w1 = PA.center:get(1)
z.t1 = PA.through:get(1)
```



Another example

File `apo_CLL3.lua`:

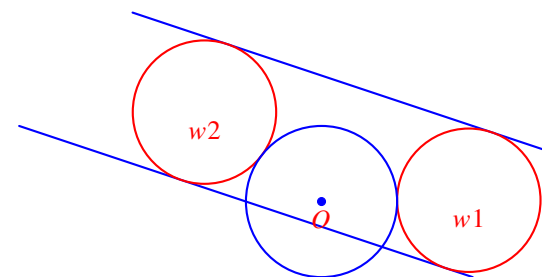
```
init_elements()
z.A = point(0, 0)
z.B = point(6, -2)
L.AB = line(z.A, z.B)
z.C = point(-3, -4)
z.D = point(3, 1)
L.CD = line(z.C, z.D)
z.O = z.D + point(3,1)
z.X = z.O + point(0,1)
C.OX = circle(z.O, z.X)
PA.center, PA.through = C.OX:CLL_all(L.AB, L.CD)
z.w1 = PA.center:get(2)
z.t1 = PA.through:get(2)
```



The last case: the lines are parallel

File apo_CLL4.lua:

```
init_elements()
z.A = point(0, 0)
z.B = point(6, -2)
L.AB = line(z.A, z.B)
z.C = point(-3, -3)
z.D = point(3, -5)
L.CD = line(z.C, z.D)
z.O = point(2, -4)
z.X = z.O + point(0, 2)
C.OX = circle(z.O, z.X)
PA.center, PA.through, n = C.OX:CLL(L.AB, L.CD)
tkz.nodes_from_paths(PA.center, PA.through)
```



31.6. Class circle: method CPP(p, p)

Circle tangent to a given circle and passing through two points.

Purpose:

Construct the circle(s) tangent to a given circle (the receiver `self`) and passing through two given points `a` and `b`.

The name CPP stands for: *circle tangent to one circle and passing through two points*.

Remarks: This method returns one or two solution circles, depending on the configuration. A valid solution exists only if both points are on the *same side* of the given circle (both inside or both outside).

Parameters:

- `a, b` — two distinct points through which the solution circles must pass.

Syntax: `local PA.center, PA.through, n = C.OX:CPP(a, b)`

Returns:

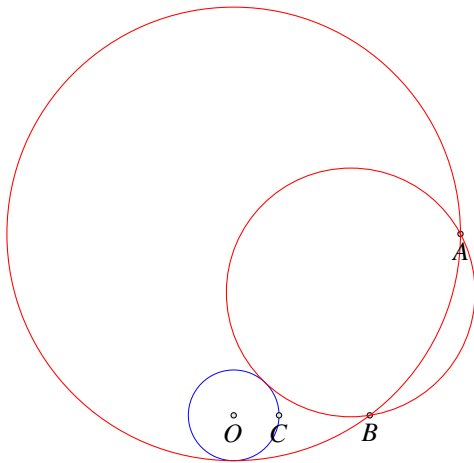
- `PA.center` — a path containing the centers of the solution circles;
- `PA.through` — a path containing the corresponding points on each circle;
- `n` — the number of solutions (0, 1, or 2).

Usage: After calling this method, the Lua function `tkz.nodes_from_paths(PA.center, PA.through)` creates the corresponding nodes `z.w1, z.w2, ...` and `z.t1, z.t2, ...` for TeX drawing. You can then draw all circles within a `tikzpicture` environment using:
`\tkzDrawCirclesFromPaths(PA.center, PA.through).`

Existence condition: A solution exists only if both points `a` and `b` are on the same side of the given circle (either both inside or both outside its disk). If one point lies inside and the other outside, no solution is returned (`n = 0`).

File apo_CPP1.lua:

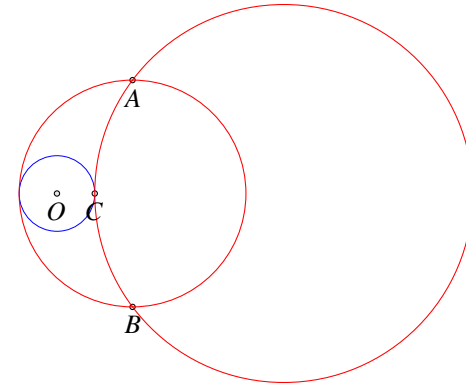
```
init_elements()
z.A = point(5,4)
z.B = point(3,0)
z.O = point(0,0)
z.C = point(1,0)
C.OC = circle(z.O, z.C)
PA.center, PA.through,
n = C.OC:CPP(z.A, z.B)
```



Special cases:
Equidistant points from the center:

File apo_CPP2.lua:

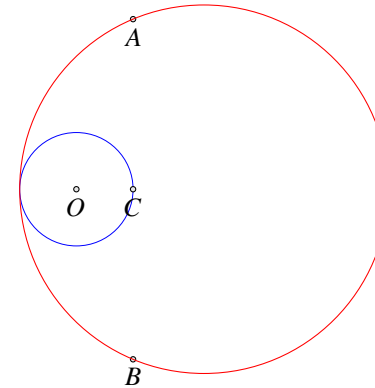
```
init_elements()
z.A = point(2,3)
z.B = point(2,-3)
z.O = point(0,0)
z.C = point(1,0)
C.OC = circle(z.O, z.C)
PA.center,
PA.through,
n = C.OC:CPP(z.A, z.B)
```



Tangent case: When the line (AB) is tangent to the initial circle, there may be a single (degenerate) solution. It can even happen that (AB) is tangent and both points are equidistant from the center.

File apo_CPP3.lua:

```
init_elements()
z.A = point(1,3)
z.B = point(1,-3)
z.O = point(0,0)
z.C = point(1,0)
C.OC = circle(z.O, z.C)
PA.center, PA.through,
n = C.OC:CPP(z.A, z.B)
```



31.7. Class circle: method CLP(L, p, mode)

Purpose: Circle tangent to a given line and to the current circle, passing through a given point.

Syntax: PA.center, PA.through, n = circle:CLP(L, p [, mode])

- L — a line object.
- p — a point through which the solution circle(s) must pass.
- mode — optional string in "all", "external", "internal". Default is "all".

Returns: three values:

- (1) `PA.center` — path of centers,
- (2) `PA.through` — path of through-points,
- (3) `n` — number of solutions (0...4).

These paths can be drawn directly with:
`\tkzDrawCirclesFromPaths(PA.center, PA.through).`
 Existence and number of solutions

- *Generic case* (point not on line nor circle): up to **4** circles.
- *Special cases* (only 2 or 0 solutions):
 - p on the circle : 2 solutions,
 - p on the line : 2 solutions,
 - line tangent to the circle : one family may collapse,
 - p at line-circle intersection : no non-trivial solution.

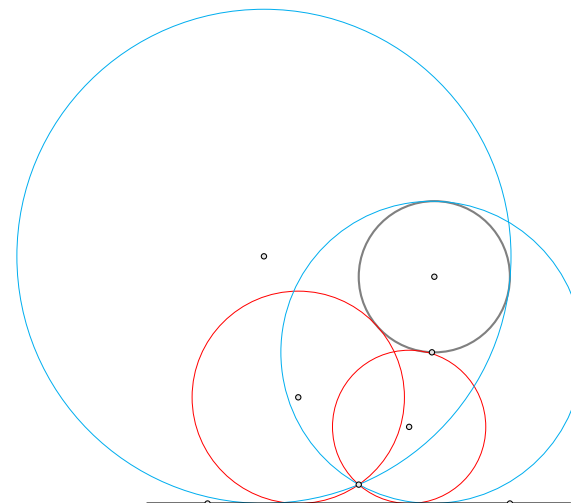
Options

- `"all"` (default) — internal and external families (up to 4),
- `"external"` — only externally tangent solutions,
- `"internal"` — only internally tangent solutions (`"inside"` accepted).

Examples usage

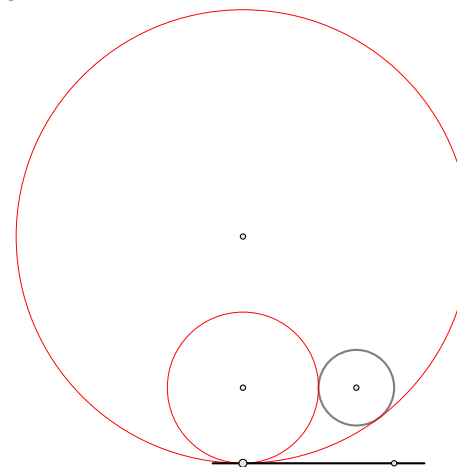
File `apo_CLP1.lua`:

```
init_elements()
z.A = point(0, 0)
z.B = point(4, 0)
L.AB = line (z.A, z.B)
z.O = point(3, 3)
z.T = point(3, 2)
z.P = point(2, .25)
C.OT = circle(z.O, z.T)
PA.center, PA.through = C.OT:CLP(L.AB, z.P)
z.O1 = PA.center:get(1)
z.O2 = PA.center:get(2)
PA.center,
PA.through = C.OT:CLP(L.AB, z.P, 'internal')
z.O3 = PA.center:get(1)
z.O4 = PA.center:get(2)
```



File `apo_CLP2.lua`:

```
init_elements()
z.A = point(0, 0)
z.B = point(4, 0)
L.AB = line (z.A, z.B)
z.O = point(3, 2)
z.T = point(3, 1)
C.OT = circle(z.O, z.T)
PA.center,
PA.through = C.OT:CLP(L.AB, z.A, "internal")
z.O1 = PA.center:get(1)
z.O2 = PA.center:get(2)
```

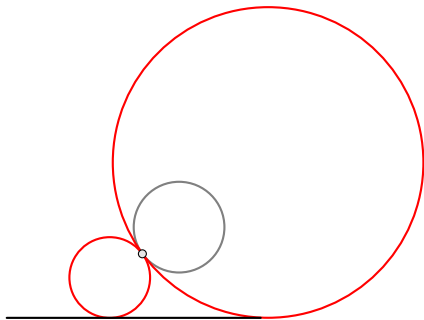


File `apo_CLP3.lua`:

```

init_elements()
z.A = point(0, 0)
z.B = point(4, 0)
L.AB = line(z.A, z.B)
z.O = point(3, 2)
z.T = point(2, 2)
C.OT = circle(z.O, z.T)
z.P = C.OT:point(0.1)
C.OT = circle(z.O, z.T)
PA.center,
PA.through = C.OT:CLP(L.AB, z.P)

```



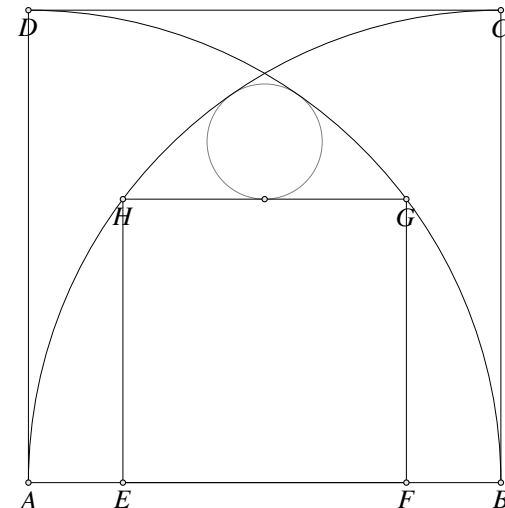
Application:

File apo_CLP4.lua:

```

init_elements()
xB = 5
z.A = point(0, 0)
z.B = point(xB, 0)
L.AB = line(z.A, z.B)
C.AB = circle(z.A, z.B)
C.BA = circle(z.B, z.A)
S.AB = L.AB:square()
_, _, z.C, z.D = S.AB:get()
z.E = point(1, 0)
z.F = point(4, 0)
L.EF = line(z.E, z.F)
S.EF = L.EF:square()
_, _, z.G, z.H = S.EF:get()
z.M = S.EF.cd.mid
PA.center, PA.through = C.AB:CLP(S.EF.cd, z.M)
z.w = PA.center:get(2)
z.t = PA.through:get(2)

```



31.8. Class circle: method CCP(C, p, mode)

Purpose: Circle tangent to two given circles and passing through a point. This method constructs 0, 2, or 4 circles depending on the configuration, passing through a given point p and tangent to two given circles.

The name CCP stands for:

circle tangent to two circles and passing through a point.

medskip Syntax: C1, C2 = C.A:CCP(C.B, p) -- external solutions
C3, C4 = C.A:CCP(C.B, p, "internal") -- internal solutions

The general problem may admit up to four solutions: two circles tangent along the *external* common tangents, and two circles tangent along the *internal* common tangents. For convenience, the method accepts an option:

- "external" (default): returns the two solutions associated with the external tangents,
- "internal": returns the two solutions associated with the internal tangents.

Special cases:

- If the point p lies strictly inside or outside both disks, up to four solutions exist in total (two external, two internal).
- If p lies on the circumference of one of the given circles, then p is already a contact point; only two solutions remain.
- If p lies simultaneously on both circles (intersection points), the configuration is degenerate and usually no non-trivial solution exists.

Examples usage:

File apo_CCP1.lua:

```

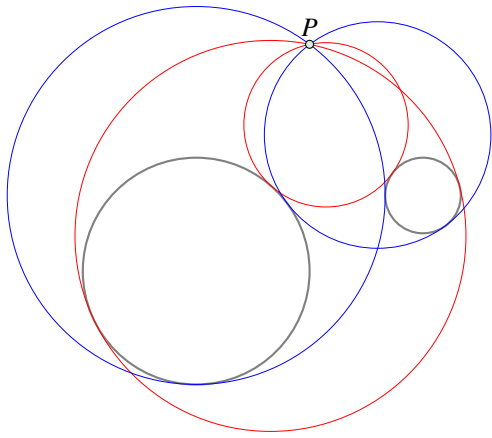
init_elements()
z.A = point(0, 0)
z.TA = point(3, 0)
z.B = point(6, 2)

```

```

z.TB = point(6, 1)
z.P = point(3, 6)
C.A = circle(z.A, z.TA)
C.B = circle(z.B, z.TB)
PA.center, PA.through = C.A:CCP(C.B,
    z.P,"external")
z.O1 = PA.center:get(1)
z.O2 = PA.center:get(2)
z.T1 = PA.through:get(1)
z.T2 = PA.through:get(2)
PA.center, PA.through = C.A:CCP(C.B,
    z.P,"internal")
z.O3 = PA.center:get(1)
z.O4 = PA.center:get(2)
z.T3 = PA.through:get(1)
z.T4 = PA.through:get(2)

```



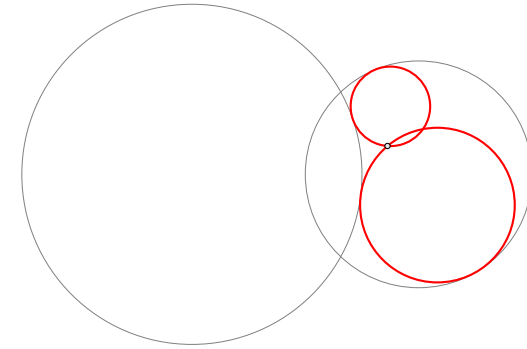
The point is inside a circle

File apo_CCP2.lua:

```

init_elements()
z.A = point(0, 0)
z.TA = point(3, 0)
z.B = point(4, 0)
L.AB = line(z.A, z.B)
z.TB = point(6, 0)
C.A = circle(z.A, z.TA)
C.B = circle(z.B, z.TB)
z.P = point(3.45, 0.5)
z.X, z.Y = intersection(C.A,C.B)
PA.center, PA.through = C.A:CCP(C.B, z.P)
tkz.nodes_from_paths(PA.center, PA.through)

```



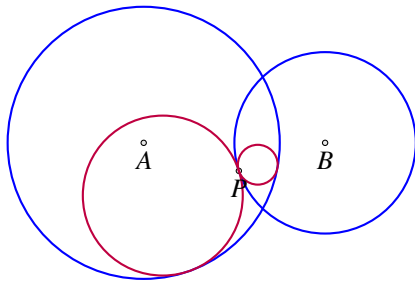
The point is on a circle

File apo_CCP3.lua:

```

init_elements()
z.A = point(0, 0)
z.a = point(3, 0)
z.B = point(4, 0)
z.b = z.B + point(0,2)
L.AB = line(z.A, z.B)
C.Aa = circle(z.A, z.a)
C.Bb = circle(z.B, z.b)
z.P = C.Bb:point(0.3)
PA.center,
PA.through,n = C.Aa:CCP(C.Bb, z.P)

```



31.9. Class circle: method CCL(C2, L)

Purpose: Find all circles that are tangent to:

- the current circle **self** (called **C1**),
- another circle **C2**,
- and a line **L**.

This method returns all possible solution circles (up to four). **Syntax:**

```
PA.center, PA.through, n = C1:CCL(C2, L)
```

- **C1** — the circle on which the method is applied (**self**),
- **C2** — another circle to which the solution circle must be tangent,
- **L** — a line object.

Returns:

- **pc** — a **path** containing the centers of the solution circles,
- **pt** — a **path** containing the corresponding tangency points on **C1**,
- **n** — the number of solutions (from 0 to 4).

Special cases.

- If one of the circles is already tangent to **L** at a point **p**, any solution circle must also pass through **p**.
- If the line **L** intersects **C1**, no circle can be tangent to both **C1** and **L** on that side.

- If all three objects are mutually tangent, the configuration is degenerate and may admit a single or no solution.

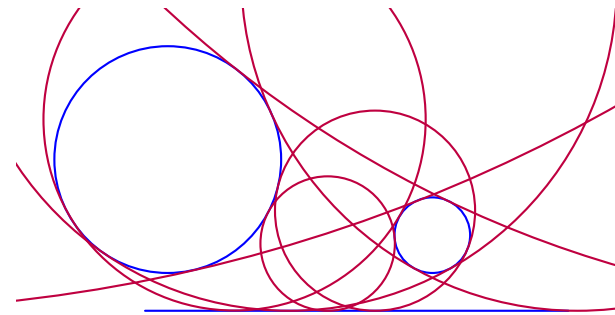
This method works with:

- `tkz.nodes_from_paths(pc, pt)` to name the solution points (**w1,t1**), (**w2,t2**), etc.,
- `\tkzDrawCirclesFromPaths(pc, pt)` to draw all the solution circles.

Example A:

File `apo_CCLA.lua`:

```
init_elements()
z.A = point(-4, 4)
z.B = z.A + point(3, 0)
C.AB = circle(z.A, z.B)
z.C = point(3,2)
z.D = z.C + point(1,0)
C.CD = circle(z.C, z.D)
z.E = point(-3, 0)
z.F = point(5, 0)
L.EF = line(z.E, z.F)
PA.center, PA.through, n = C.AB:CCL(C.CD, L.EF)
tkz.nodes_from_paths(PA.center, PA.through)
```



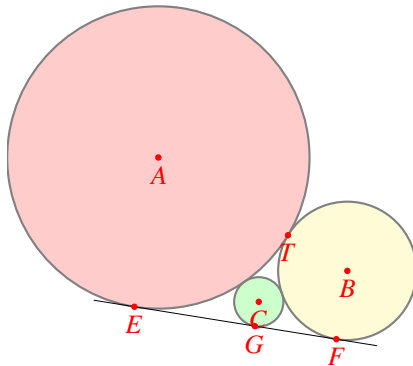
Example B:

File `apo_CCLB.lua`:

```

init_elements()
z.A = point(1, 5)
z.B = point(6, 2)
C.A = circle(from_radius(z.A, 4))
L.AB = line(z.A, z.B)
z.T = intersection(L.AB, C.A, {near = z.B})
C.B = circle(z.B, z.T)
L.TG1, L.TG2 = C.A:common_tangent(C.B,"external")
z.E, z.F = L.TG2:get()
PA.center, _, _ = C.A:CCL(C.B, L.TG2)
z.C = PA.center:get(1)
z.G = L.TG2:projection(z.C)

```



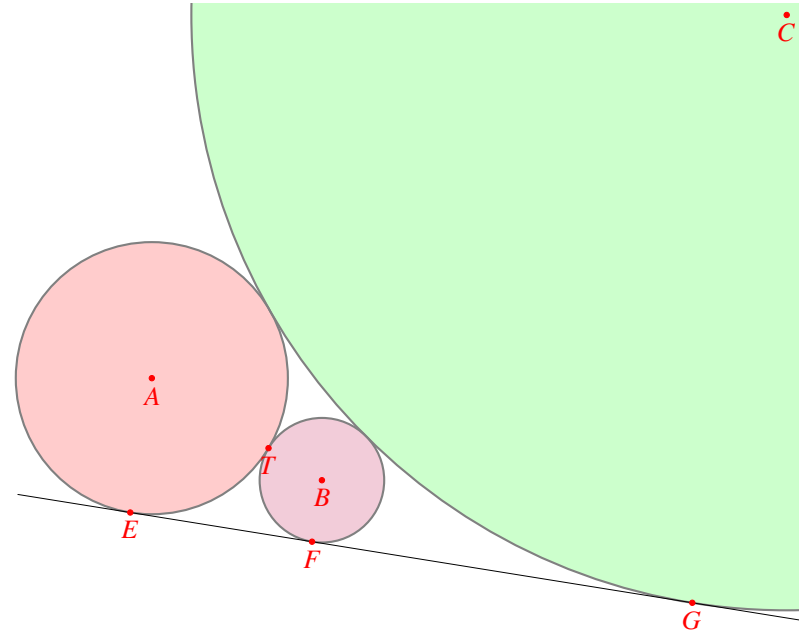
Example C:

File apo_CCLC.lua:

```

init_elements()
z.A = point(1, 5)
z.B = point(6, 2)
C.A = circle(from_radius(z.A, 4))
L.AB = line(z.A, z.B)
z.T = intersection(L.AB, C.A, {near = z.B})
C.B = circle(z.B, z.T)
L.TG1, L.TG2 = C.A:common_tangent(C.B,"external")
z.E, z.F = L.TG2:get()
PA.center, _, _ = C.A:CCL(C.B, L.TG2)
z.C = PA.center:get(2)
z.G = L.TG2:projection(z.C)

```



31.10. Case circle: method CCC(C2, C3 [, opts])

Purpose: Circles tangent to three given circles $C_1 = \text{self}$, C_2 , C_3 (up to 8 solutions).
Implementation follows Viète's reduction.

Syntax: `pc, pt, n = C1:CCC(C2, C3 [, opts])`

- C_1 — the current circle (`self`).
- C_2, C_3 — other circle objects.
- `opts` — optional table of tolerances:
 - `opts.abs` (default $1e-4$) — absolute tolerance.
 - `opts.rel` (default $1e-6$) — relative tolerance.

Returns:

- (1) `pc` — path of solution centers,
- (2) `pt` — path of corresponding “through” points,
- (3) `n` — number of solutions found (0 to 8).

Result use in TikZ.

- With global paths: store `PA.center`, `PA.through`, then draw all circles using:
`\tkzDrawCirclesFromPaths(PA.center, PA.through)`.
- With local paths: use `\tkz.nodes_from_paths(pc, pt)` to create nodes `w1,t1,w2,t2`,
...

Example A — Using global paths

```

\directlua{
  PA.center, PA.through, tkzNbCircles = C.A:CCC(C.B, C.C)
}

```

```
\tkzDrawCirclesFromPaths(PA.center, PA.through)
```

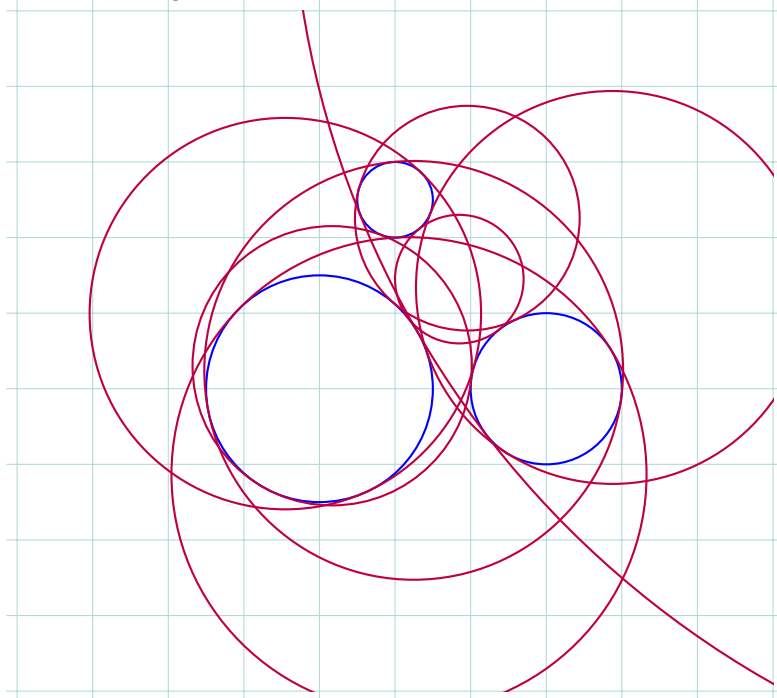
Example B — Using local paths

```
\directlua{
  local pc, pt, n = C.A:CCC(C.B, C.C)
  tkz.nodes_from_paths(pc, pt)
}
% circles are now (w1,t1), (w2,t2), ...
```

Example C

File apo_CCCC.lua:

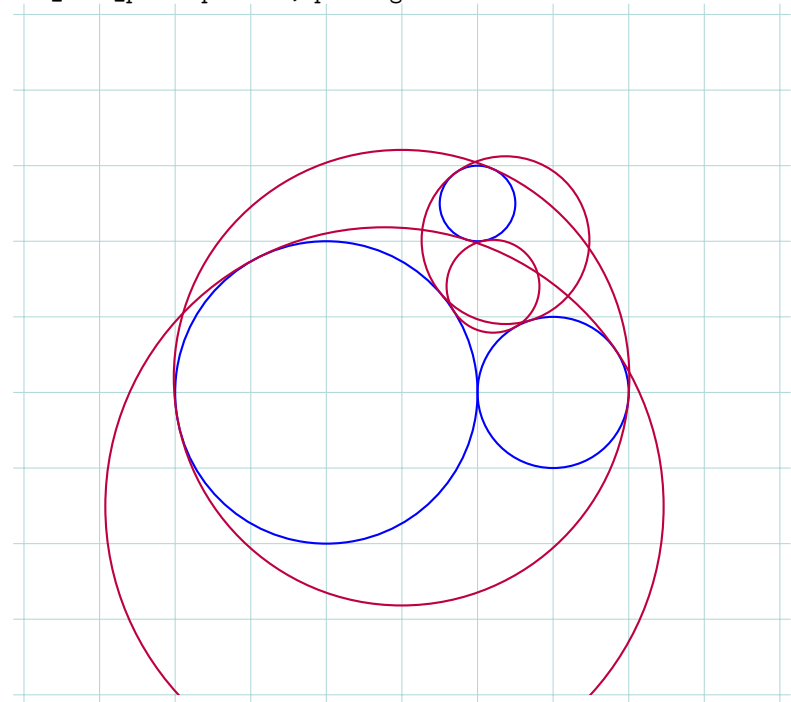
```
init_elements()
z.A = point(0, 0)
z.B = point(6, 0)
z.C = point(2, 5)
z.a = z.A + point(3, 0)
z.b = z.B + point(2, 0)
z.c = z.C + point(1, 0)
C.Aa = circle(z.A, z.a)
C.Bb = circle(z.B, z.b)
C.Cc = circle(z.C, z.c)
PA.center, PA.through, n = C.Aa:CCC(C.Bb, C.Cc)
```



Example D

File apo_CCCD.lua:

```
init_elements()
z.A = point(0, 0)
z.B = point(6, 0)
z.C = point(4, 5)
z.a = z.A + point(4, 0)
z.b = z.B + point(2, 0)
z.c = z.C + point(1, 0)
C.Aa = circle(z.A, z.a)
C.Bb = circle(z.B, z.b)
C.Cc = circle(z.C, z.c)
pcenter, pthrough, n = C.Aa:CCC(C.Bb, C.Cc)
tkz.nodes_from_paths(pcenter, pthrough)
```



Example E

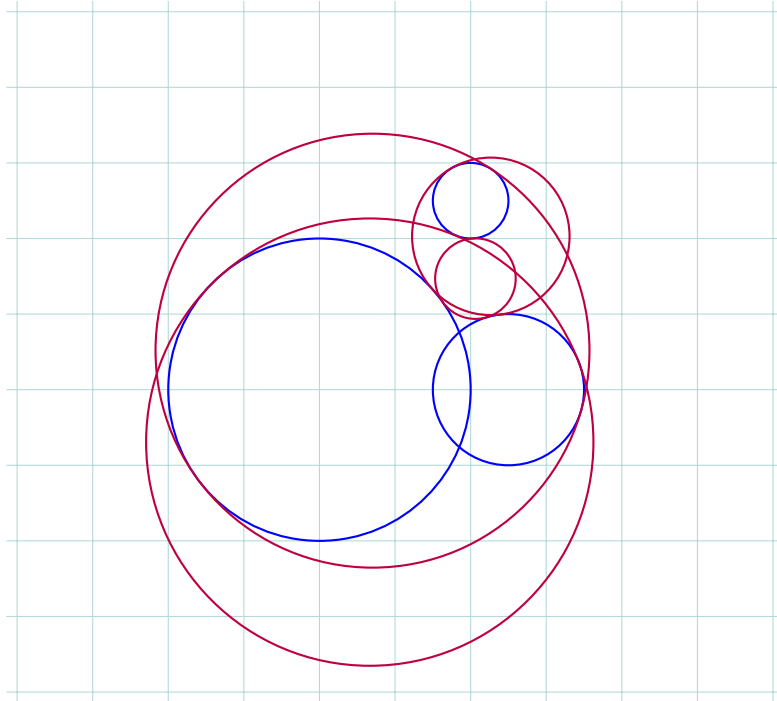
File apo_CCCE.lua:

```
init_elements()
z.A = point(0, 0)
z.B = point(5, 0)
z.C = point(4, 5)
z.a = z.A + point(4, 0)
z.b = z.B + point(2, 0)
z.c = z.C + point(1, 0)
C.Aa = circle(z.A, z.a)
```

```

C.Bb = circle(z.B, z.b)
C.Cc = circle(z.C, z.c)
PA.center, PA.through, n = C.Aa:CCC(C.Bb, C.Cc)

```



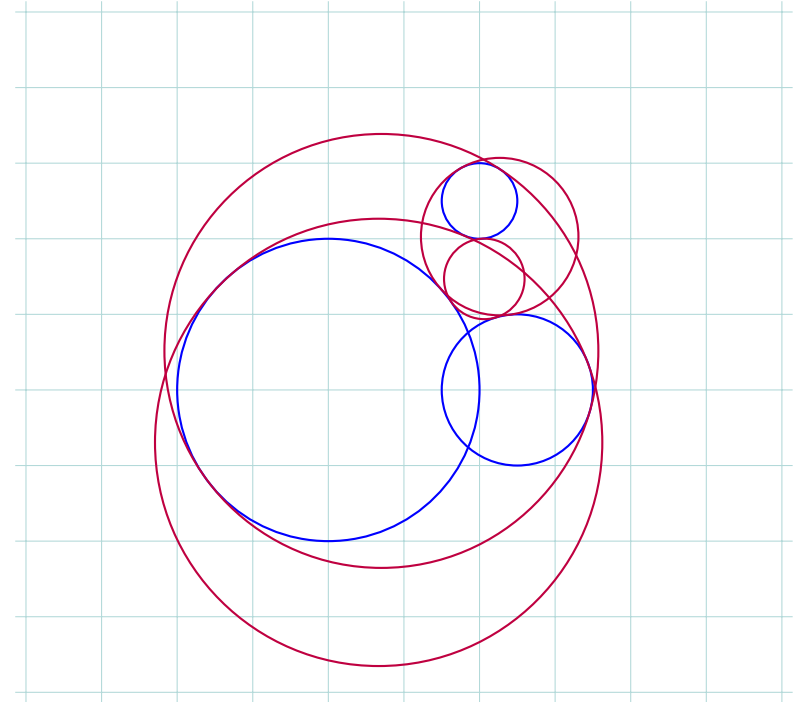
Example F

File apo_CCCE.lua:

```

init_elements()
z.A = point(0, 0)
z.B = point(5, 0)
z.C = point(4, 5)
z.a = z.A + point(4, 0)
z.b = z.B + point(2, 0)
z.c = z.C + point(1, 0)
C.Aa = circle(z.A, z.a)
C.Bb = circle(z.B, z.b)
C.Cc = circle(z.C, z.c)
PA.center, PA.through, n = C.Aa:CCC(C.Bb, C.Cc)

```



31.11. Class circle: method CCC_gergonne(C_2, C_3)

Purpose: Construct all circles tangent to three given circles using the classical Gergonne construction: centers of similitude, radical axis, poles, and contact points. This method is only valid for circles that are disjoint in pairs.

Syntax:

PA.center, PA.through, n = C1:CCC_gergonne(C2, C3 [, opts])

Arguments:

- C1 — the calling circle (self).
- C2, C3 — two other circles.
- opts.abs — optional absolute tolerance for tangency tests (default = tkz.epsilon).

Returns:

- PA.center — a path storing the centers of all solution circles,
- PA.through — a path storing one contact point on each solution circle,
- n — the number of solutions found (may range from 0 to 8).

Each circle solution is represented by the pair:

$(PA.center[i], PA.through[i]), \quad i = 1, \dots, n.$

It can be drawn using

`\tkzDrawCirclesFromPaths(PA.center, PA.through).`

Geometric principle: The construction follows Gergonne:

- (1) Compute the external and internal centers of similitude of the three circles: J_{12}, J_{13}, J_{23} (external), I_{12}, I_{13}, I_{23} (internal).

- (2) These six points are aligned on four special lines: one line through the three external centers (“JJJ”), and three lines passing through one external and the two internal centers (“JII”).
- (3) The radical center R of the three circles is computed (intersection of two radical axes).
- (4) For each of the four lines, compute its pole with respect to each circle. From R , draw lines to these poles; each such line intersects each circle (in general) in two points: these are candidate contact points.
- (5) For each triple of contact points (one on each circle), compute the circumcenter. This is a potential solution. Keep it only if it is tangent to all three circles (checked by `circle:act`).

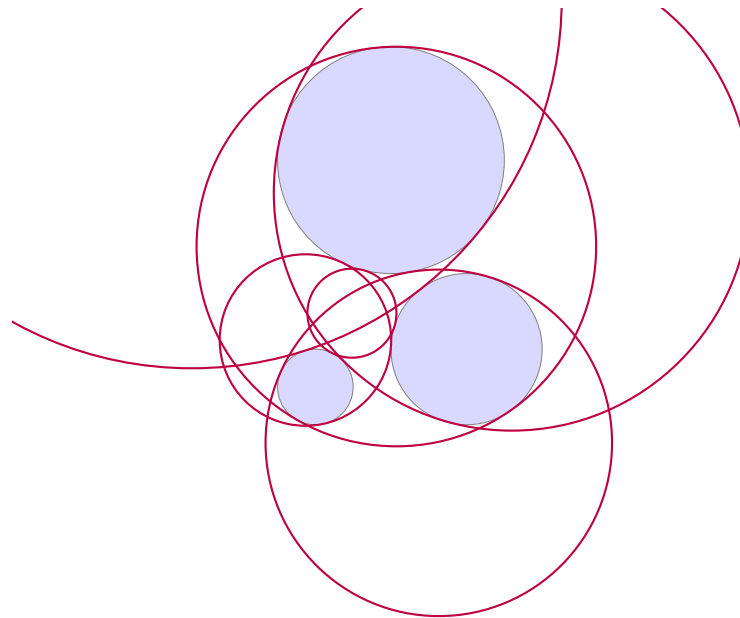
Remarks:

- This method is purely geometric and works well when the circles are disjoint or only tangent. If the three circles intersect (secant configuration), some poles or intersections may be undefined, and no solution is returned.
- The method complements more robust algebraic approaches (Viète-based solvers).
- The returned paths can be directly used in a TikZ picture.

Example:

File `apo_gergonne.lua`:

```
init_elements()
z.A = point(0, 0)
z.a = z.A + point (1, 0)
z.B = point (4, 1)
z.b = z.B + point (2, 0)
z.C = point (2, 6)
z.c = z.C + point (3, 0)
C.Aa = circle(z.A, z.a)
C.Bb = circle(z.B, z.b)
C.Cc = circle(z.C, z.c)
PA.center, PA.through = C.Aa:CCC_gergonne(C.Bb, C.Cc)
```



Explanations:

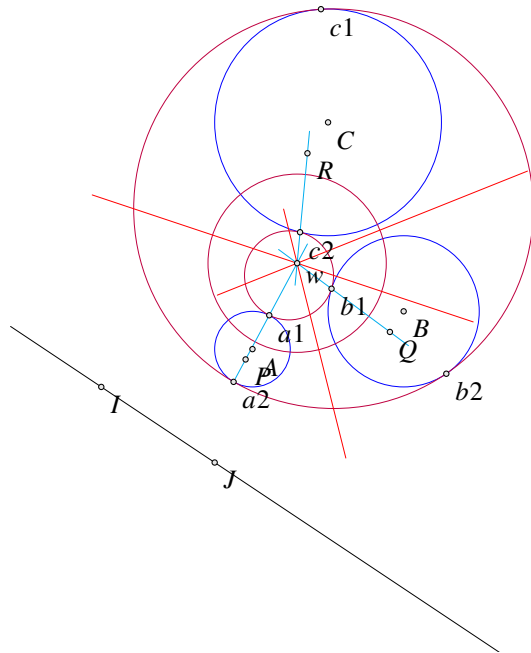
File `apo_gergonne2.lua`:

```
init_elements()
z.A = point(0, 0)
z.a = z.A + point (1, 0)
z.B = point (4, 1)
z.b = z.B + point (2, 0)
z.C = point (2, 6)
z.c = z.C + point (3, 0)
C.Aa = circle(z.A, z.a)
C.Bb = circle(z.B, z.b)
C.Cc = circle(z.C, z.c)
L.rAB = C.Aa:radical_axis (C.Bb)
L.rAC = C.Aa:radical_axis (C.Cc)
L.rBC = C.Bb:radical_axis (C.Cc)
z.w = C.Aa:radical_center(C.Bb, C.Cc)
z.x1, z.y1 = L.rAB:get()
z.x2, z.y2 = L.rAC:get()
z.x3, z.y3 = L.rBC:get()
z.t = C.Aa:radical_circle(C.Bb, C.Cc).through
z.I = C.Aa:external_similitude(C.Bb)
z.J = C.Aa:external_similitude(C.Cc)
z.K = C.Bb:external_similitude(C.Cc)
```

```

L.IK = line(z.I, z.K)
z.P = C.Aa:pole(L.IK)
z.Q = C.Bb:pole(L.IK)
z.R = C.Cc:pole(L.IK)
z.a1, z.a2 = intersection_lc_(z.w, z.P, z.A, z.a)
z.b1, z.b2 = intersection_lc_(z.w, z.Q, z.B, z.b)
z.c1, z.c2 = intersection_lc_(z.w, z.R, z.C, z.c)
z.w1 = circum_center_(z.a1, z.b1, z.c2)
z.w2 = circum_center_(z.a2, z.b2, z.c1)

```



References

- [1] Alain Matthes. *tkz-elements*. Version 6.00c. 2026. URL: <https://ctan.org/pkg/tkz-elements> (visited on 10/11/2026).
- [2] Evan Chen. *Lemmas in AoPS Geometry*. 2023. URL: <https://web.evanchen.cc/handouts/GeoSlang/GeoSlang.pdf> (visited on 01/23/2026).
- [3] Brilliant.org. *Pascal's Theorem*. 2024. URL: <https://brilliant.org/wiki/pascals-theorem/> (visited on 01/23/2026).
- [4] Alain Matthes. *tkz-euclide*. Version 5.13c. 2026. URL: <https://ctan.org/pkg/tkz-euclide> (visited on 01/25/2026).
- [5] Eric W. Weisstein. *Apollonius Circle*. From *MathWorld—A Wolfram Web Resource*. URL: <https://mathworld.wolfram.com/ApolloniusCircle.html> (visited on 01/23/2026).
- [6] Eric W. Weisstein. *Tangent Circles*. From *MathWorld—A Wolfram Web Resource*. URL: <https://mathworld.wolfram.com/TangentCircles.html> (visited on 01/23/2026).