

Alain Matthes

March 11, 2026 Documentation V.1.00

<http://altermundus.fr>

Golden Arbelos

Alain Matthes

This document presents some properties of the arbelos built around the divine proportion. Most of the demonstrations have been found on the Internet. I have quoted the authors when possible. All figures were obtained with my packages [tkz-elements](#) and [tkz-euclide](#).

☞ You will find most of the examples in this document and many other examples on my website : altermundus.fr.

Please report typos or any other comments to this documentation to [Alain Matthes](#)

Contents

List of Figures

1 Presentation

In this paper I will focus on the different constructions of the first Pappus circle in an arbelos, indicating which geometrical properties one can note. In the particular case of arbelos built from a divine proportion, I will indicate which special constructions can be used as well as which geometrical peculiarities result from it.

1.1 Definition

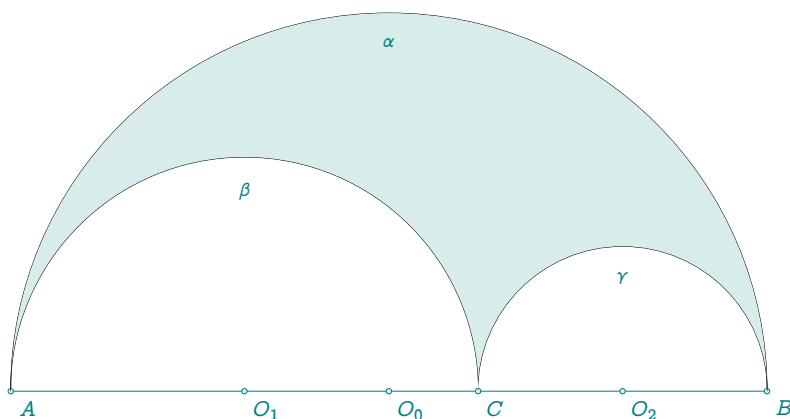


Figure 1: Arbelos

Arbelos Definition by Archimedes' Book of Lemmas

If $[AB]$ be the diameter of a semicircle and C any point on $[AB]$, and if semicircles be described within the first semicircle and having $[AC]$, $[CB]$ as diameters respectively, the figure included between the circumferences of the three semicircles is called *arbelos*. Let β, γ, α be the semicircles of respective diameters $[AC]$, $[CB]$ and $[AB]$, then the arbelos bounded by the three semicircles is denoted (β, γ, α) or (A, C, B) .

In the rest of the document, we will refer to a , b et $a + b$ the respective radii of the semicircles β , γ and α . O_1 , O_2 and O_0 will be the respective centers of the semicircles.

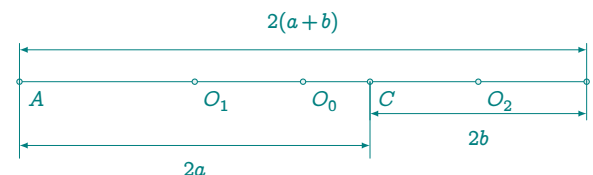


Figure 2: notations

1.2 About the Devine Proportion

1.2.1 Definition

Golden Ratio

In mathematics, expressed algebraically, for quantities a and b with $a > b > 0$,

$$\frac{a+b}{a} = \frac{a}{b} = \varphi$$

where the Greek letter φ represents the **golden ratio** or **divine proportion**. In other words, two quantities are in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities. [wikipedia]

In the package `tkz-euclide`, you can use the macro `\tkzPhi` to get the value of golden ratio: $\varphi \approx 1.618034$.

We saw above that the Golden Ratio has the property:

$$\frac{a+b}{a} = \frac{a}{b}$$

We can split the first fraction

$$\frac{a}{a} + \frac{b}{a} = \frac{a}{b}$$

Either $1 + \frac{1}{\varphi} = \varphi$ with $\varphi = \frac{a}{b}$

Also φ is a positive solution of :

$$x^2 - x - 1 = 0$$

That is to say $\varphi = \frac{1 + \sqrt{5}}{2}$

1.2.2 Algebraic Properties

| | Formula |
|---------------------------------|--|
| - Powers of φ | $\varphi^2 = \varphi + 1$ |
| | $\varphi^3 = \varphi^2 + \varphi = 2\varphi + 1$ |
| | $\varphi^4 = 3\varphi + 2$ |
| | $\varphi^n = \varphi^{(n-1)} + \varphi^{(n-2)}$ |
| | Formula |
| - Powers of $\frac{1}{\varphi}$ | $\frac{1}{\varphi} = \varphi - 1$ |
| | $\frac{1}{\varphi^2} = 2 - \varphi = 1 - \frac{1}{\varphi}$ |
| | $\frac{1}{\varphi^3} = 2\varphi - 3 = \frac{2}{\varphi} - 1$ |

1.2.3 Geometric Properties

- **Golden section of a line segment** This is the simplest example of the golden ratio in relation to geometry. It's a special line segment portion of a line segment, called the golden segment, illustrated here:



Figure 3: Golden Segment

We consider the segment $[AB]$ and we decide to define a point C , situated between A and B in order to have:

$$\frac{AC}{CB} = \frac{AB}{AC}$$

The point C defines a divine proportion on the segment $[AB]$. AC is the golden section of the line segment $[AB]$.

It's easy to get C Just use the macro of `tkz-euclide`: `\tkzDefGoldenRatio(A,B)\tkzGetPoint{C}`.

With $AC = 2a, CB = 2b$ $a > b$

We transformed this relation by doing a cross product :

$$AC^2 = AB \cdot CB$$

which becomes

$$4a^2 = 2(a + b) \cdot 2b$$

That we can write:

$$a^2 = ab + b^2 \text{ or } \frac{a}{a} + \frac{b}{a} = \frac{a}{b}$$

- **lemma:** *The midpoints of the arcs of an arbelos is aligned.*

To help you find the midpoint of an arc, I created the macro `\tkzDefMidArc` so `\tkzDefMidArc(O,A,B)` gives the midpoint of the arc \widehat{AB} on the circle with center O .

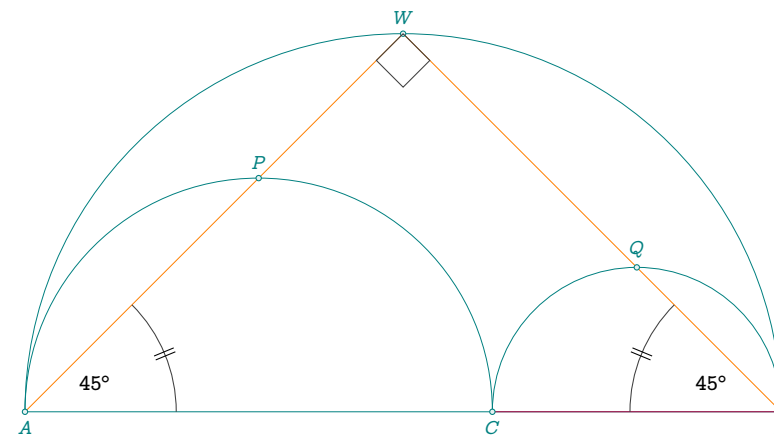


Figure 4: Midpoints of the arcs

$$3. JE = \frac{\sqrt{3}}{2}AB \text{ so } \frac{JE}{GK} = \frac{4\sqrt{3}}{2\sqrt{15}} = \frac{2}{\sqrt{5}}$$

$$4. \frac{JC}{CK} = \frac{JE}{GK} \text{ and therefore } JC = \frac{2}{\sqrt{5}} \times (JK - JC). \text{ It follows that } JC = \frac{2}{\sqrt{5}+2}JK = \frac{(\sqrt{5}-2)}{2}AB$$

$$5. AC = AJ + JC = \frac{AB}{2} + \frac{(\sqrt{5}-2)}{2}AB = \frac{(\sqrt{5}-1)}{2}AB = \frac{2}{\sqrt{5}+1}AB$$

$$6. \text{ Finally } \frac{AB}{AC} = \varphi$$

Remark: It is possible to obtain the point G by drawing the circle of center O passing through B .

– Golden Rectangle

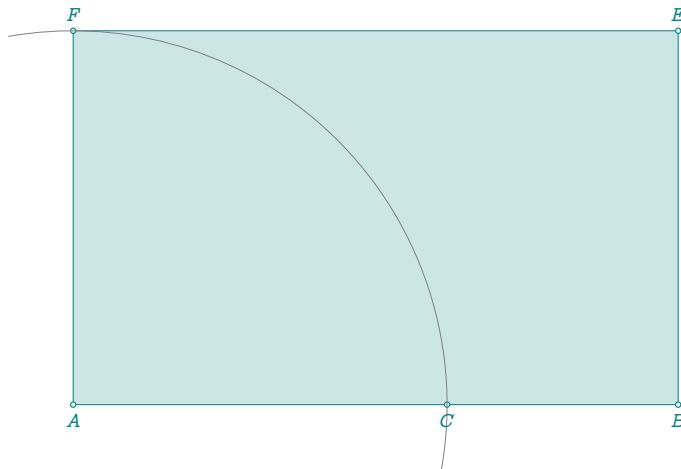


Figure 7: Golden Rectangle

The rectangle $ABEF$ is a golden rectangle because the ratio of the length AB to the width BE is equal to the golden ratio φ . The construction is easy with `tkz-euclide` and its macro

`\tkzDefGoldRectangle(A,B)`.

It is of course possible to use the previous construction to build this rectangle with the compass. C divides $[AB]$ with the divine proportion and then $AF = AC$

– Golden Triangle or Sublime Triangle

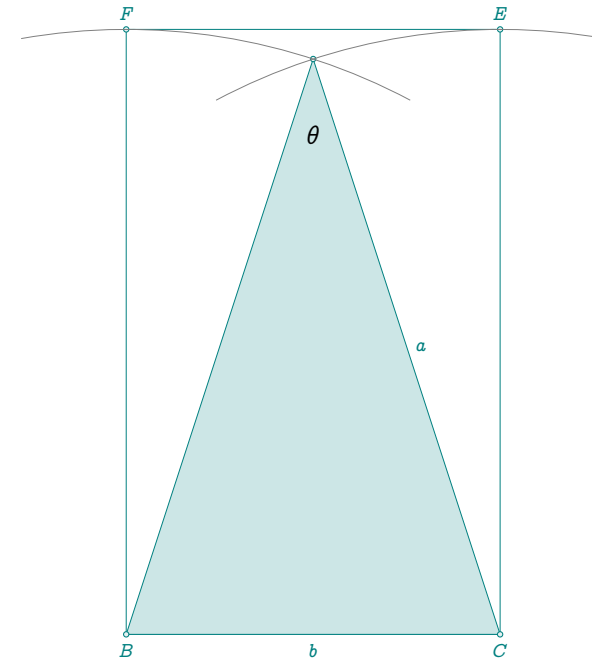


Figure 8: Golden triangle

The sublime triangle (figure ??) can be obtained from the golden rectangle. It is also possible to use a macro of `tkz-euclide`:

`\tkzDefTriangle[golden](B,C)`
`\tkzGetPoint{A}`.

In the figure (??) we have $\frac{a}{b} = \varphi$ and $\theta = 36^\circ$

- **Kepler' Triangle** A Kepler triangle is a special right triangle with edge lengths in geometric progression (Like ABC in the figure ??). The ratio of the progression is $\sqrt{\varphi}$, where φ is the golden ratio, and the progression can be written: $1:\sqrt{\varphi}:\varphi$.

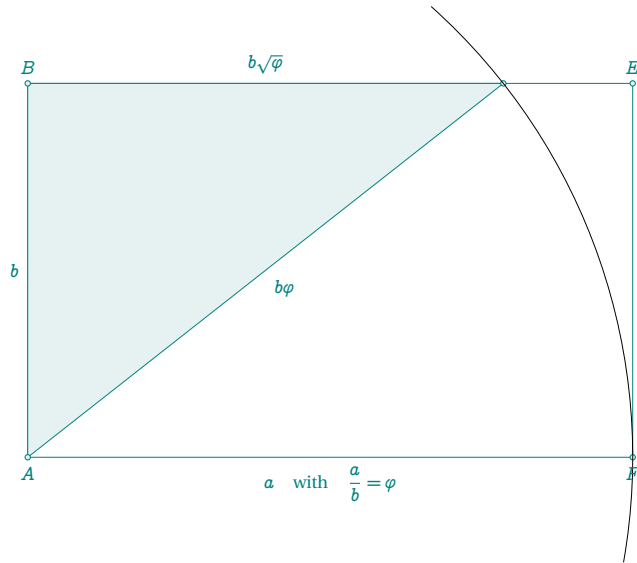


Figure 9: Kepler' Triangle

- Right triangle and golden ration

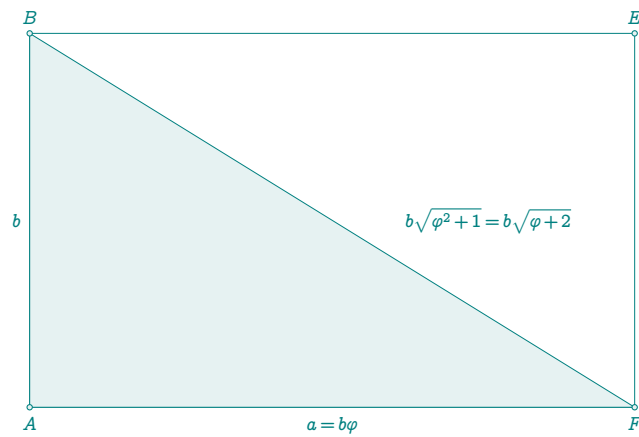


Figure 10: Right Triangle

1.3 Golden Arbelos

Definition

Let β, γ, α be the semicircles with the respective radii $a, b, a + b$. If the positive numbers a and b ($a > b$) are in a golden ratio in other words if

$$\frac{a+b}{a} = \frac{a}{b} = \varphi$$

then this special type of arbelos is called golden arbelos.

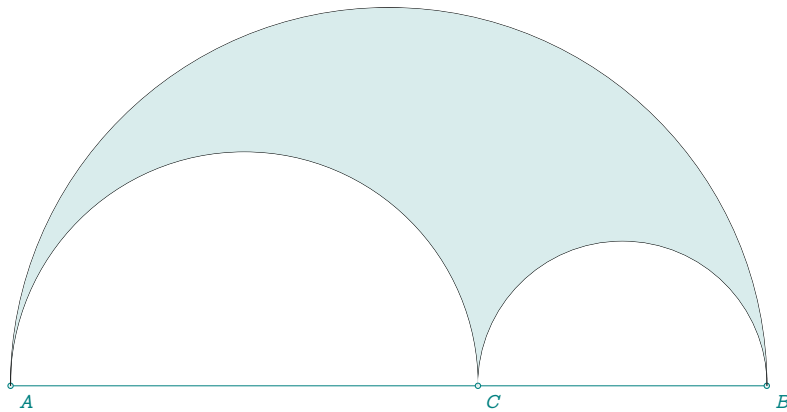


Figure 11: Golden Arbelos $\frac{AC}{CB} = \frac{AB}{AC} = \varphi$

1.3.1 Construction of the Golden Arbelos

We can consider two cases. Suppose we want to obtain the arbelos (A, C, B) then we have to examine the construction from the couple of points (A, C) or from the couple (A, B) . We will use the construction seen here ??

With (A, C)

1. The square based on (A, C) is constructed,
2. Then the middle m of $[A, C]$ is constructed,
3. The arc of center m passing through c intersects the half-line $[A, C)$ at x .

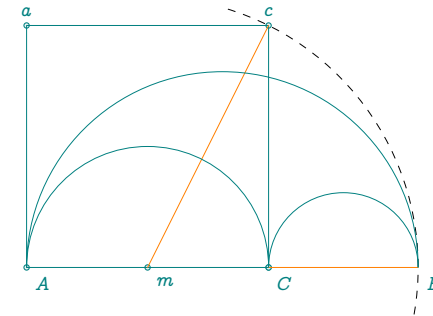


Figure 12: With A and C

With (A, B)

1. The square based on (A, B) is constructed,
2. Then the middle m of $[B, b]$ is constructed,
3. The arc of center m passing through A intersects the half-line $[b, B)$ at x ,
4. Finally the line (a, x) cuts the segment $[AB]$ at C .

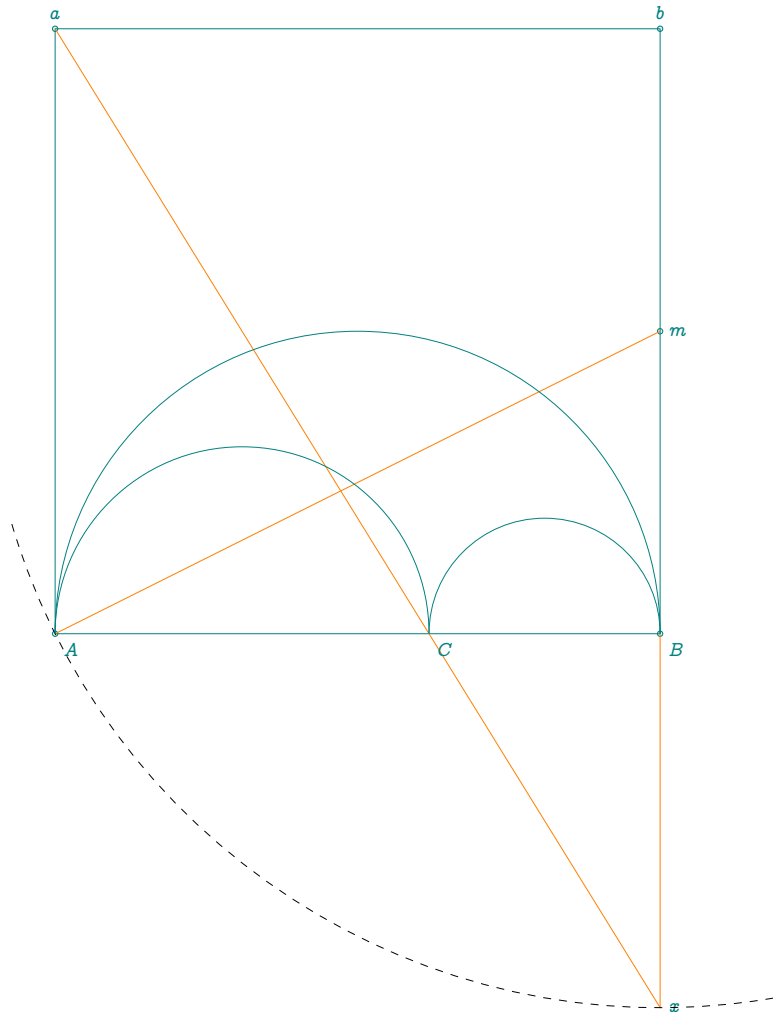


Figure 13: With A and B

1.3.2 Find the golden ratio

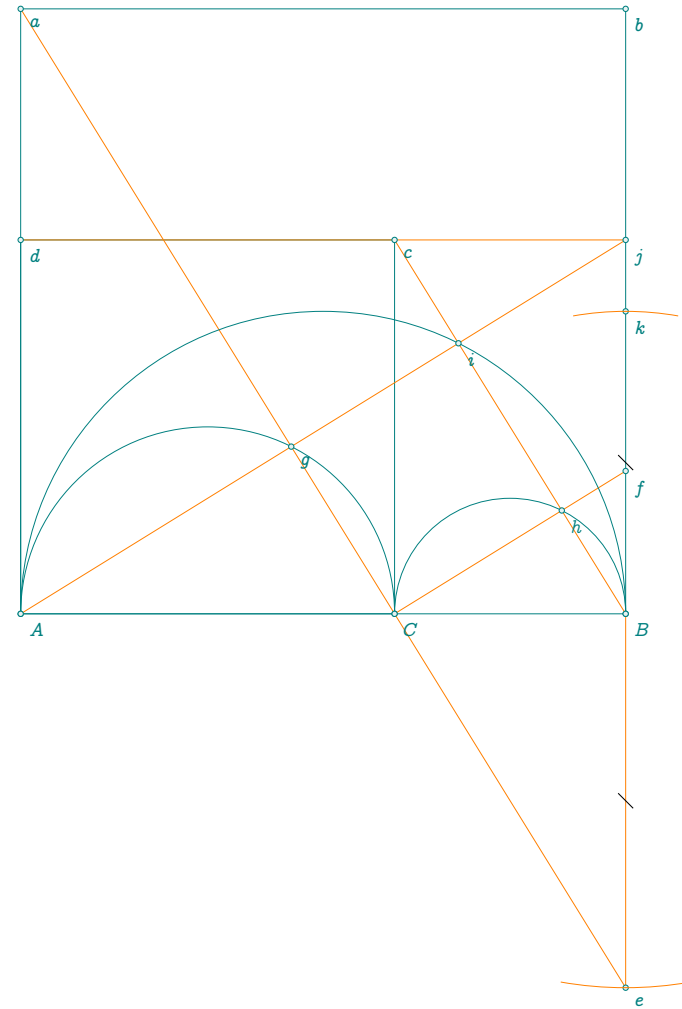
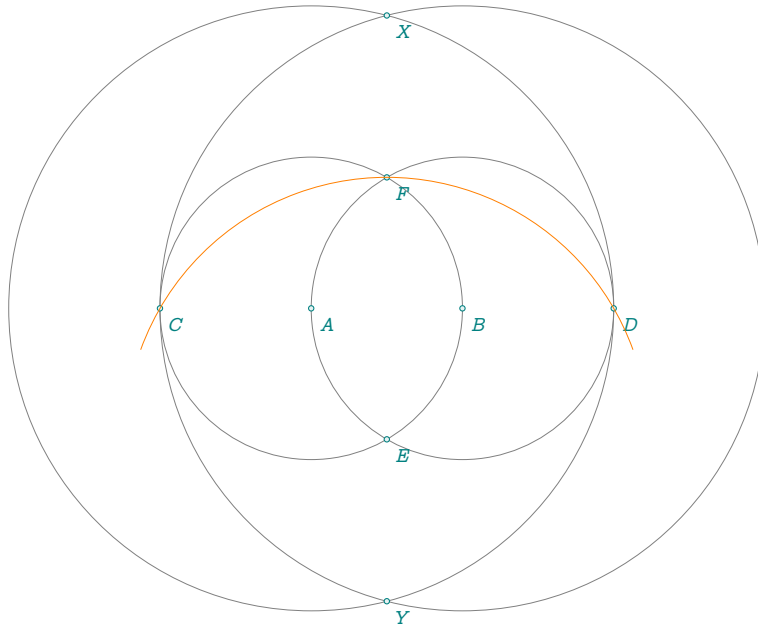


Figure 14: Some devine ratios

$$\frac{AB}{AC} = \frac{Ai}{Ag} = \frac{Bj}{fj} = \frac{Bi}{hi} = \frac{Cf}{Ch} = \varphi$$

1.3.3 Golden section with compass only

Golden section



$$\frac{XE}{EF} = 1.6180339887499 \approx$$

2 Pappus circle

2.1 First Pappus circle

Definition

Given an arbelos, there is a circle (δ) included in this arbelos and tangent to (α) , (β) and (γ) ; it is called inscribed circle or circle of Pappus. Archimedes also determined the radius of this circle.

The radius of the Pappus circle is $r = \frac{ab(a+b)}{a^2+ab+b^2}$.

Let us first analyze constructions of this circle in the general case, then we will examine the particular case with the golden arbelos.

Given a golden arbelos $(A, C, B) = (\alpha, \beta, \gamma)$, we now that (β) and γ are external tangent at C and internal tangent at (α) at A, B . Construct the circle (δ) external tangent at $((\alpha), (\beta), (\gamma))$. Let us note P_0, P_1, P_2 the points of tangency.

2.1.1 Construction of Pappus circle with arcs version 1

See paragraph [??] for a justification and the document "Simple Constructions of the Incircle of an Arbelos" [Ref: ??].

Without justifications (they will be given later), here is one of the simplest constructions.

1. An arbelos is given the semicircles (α) , (β) and (γ) have centers O_0, O_1 and O_2 . P and Q are the north poles of (β) and (γ) .
2. Two arcs of circles of respective centers P and Q passing through C determine on the arbelos the points of tangency of the circle (δ) with the arbelos: P_0, P_1 and P_2 .
3. The center of the Pappus circle is given by the intersection of the lines (O_1P_1) and (O_2P_2)

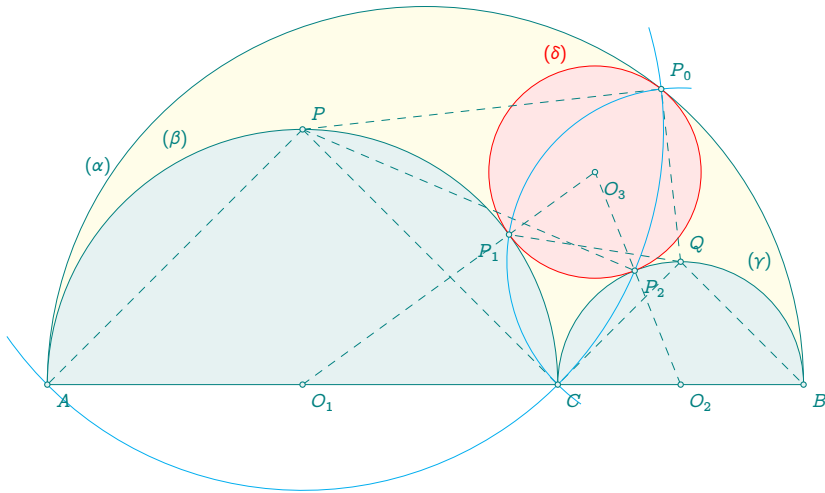


Figure 15: Pappus circle with arcs

2.1.2 Construction of Pappus circle with arcs version 2

See paragraph [??] for a justification and the document "Simple Constructions of the Incircle of an Arbelos" [Ref: ??].

Let O be the midpoint of the arc (\widehat{AB}) . Construct

- i) The line (OC) to intersect the semicircle (α) at P_0 ;
- ii) The circle, center O , through A and B , to intersect the semicircles (β) and (γ) at P_1 and P_2 .

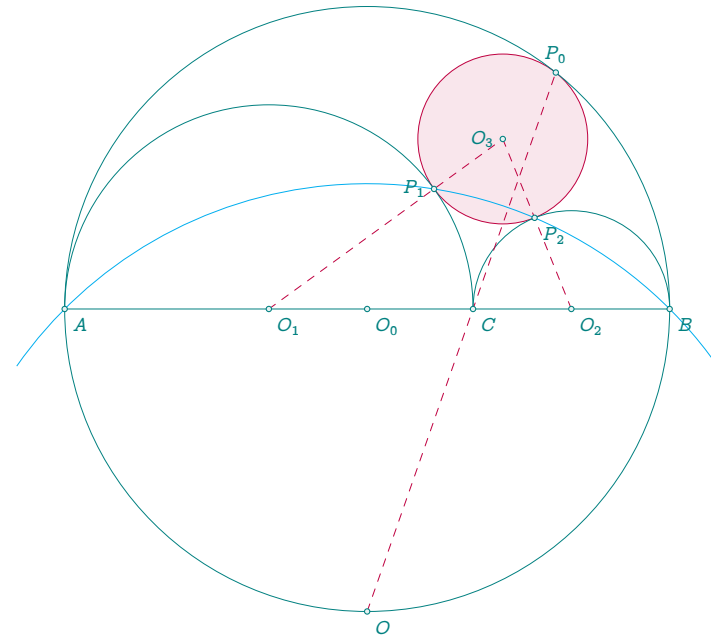


Figure 16: Pappus circle with arcs version 2

2.1.3 Construction of Pappus circle with squares

See the document "Simple Constructions of the Incircle of an Arbelos" [Ref: ??].

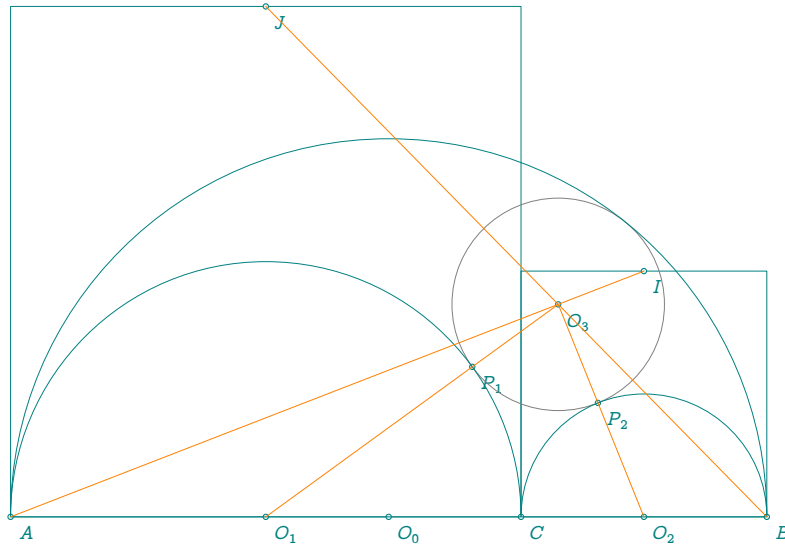


Figure 17: Pappus circle with squares

2.2 Determining the radius

Let O_3 be the center of (δ) and H_4 its orthogonal projected onto the line (AB) . H_4 belongs to the segment $[CO_2]$ since we have assumed $a > b$. From the right-angled triangles $O_0O_3H_4$, $O_1O_3H_4$ and $O_2O_3H_4$ we get

$$O_3O_0^2 = O_3H_4^2 + O_0H_4^2$$

$$O_3O_1^2 = O_3H_4^2 + O_1H_4^2$$

$$O_3O_2^2 = O_3H_4^2 + O_2H_4^2$$

Either

$$(a + b - r)^2 = O_3H_4^2 + O_0H_4^2$$

$$(a + r)^2 = O_3H_4^2 + (b + O_0H_4)^2$$

$$(b + r)^2 = O_3H_4^2 + (a - O_0H_4)^2$$

because

$$O_1H_4 = O_1O_0 + O_0H_4 = O_0H_4 + (AO_0 - AO_1) = O_0H_4 + (a + b) - a = b + O_0H_4$$

and

$$O_2H_4 = O_0O_2 - O_0H_4 = O_1O_2 - O_0O_2 - O_0H_4 = (a + b) - b - O_0H_4 = a - O_0H_4.$$

From the first two equalities we get

$$b \cdot O_0H_4 = 2ar - b^2 - ab + br$$

and from the first and third we get

$$a \cdot O_0H_4 = a^2 + ab - ar - 2br. \text{ Hence } r = \frac{ab(a + b)}{a^2 + ab + b^2}.$$

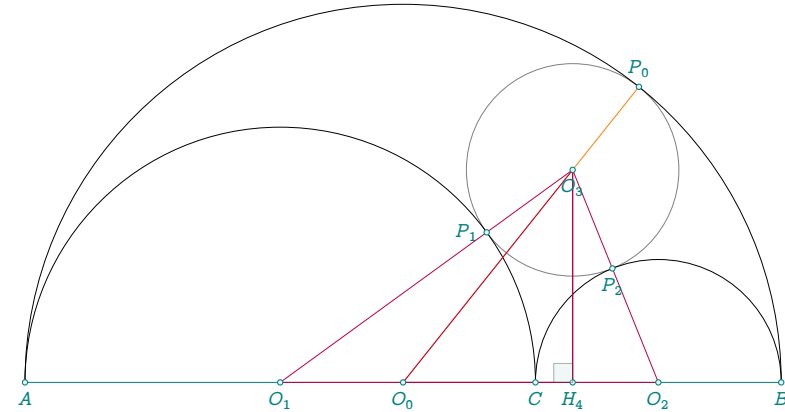


Figure 18: Pappus circle radius

Question

Let $[AB]$, the diameter of a semicircle, be divided at C so that $AB/AC = k$. Describe semicircles within the first semicircle and on $[AC]$, $[BC]$ as diameters, and suppose a circle drawn touching the all three semicircles. If $[GH]$ be the diameter of this circle, to find relation between GH and AB .

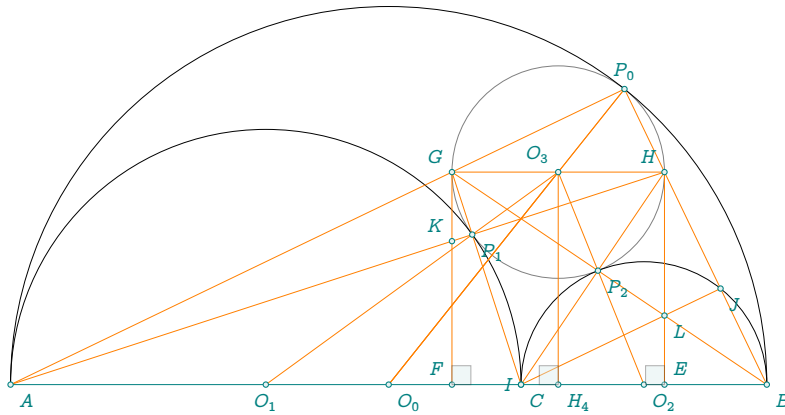


Figure 19: Pappus circle diameter

Proof:

Let $[GH]$ be the diameter of the circle which is parallel to (AC) , and let the circle touch the semicircles on $[AC]$, $[AB]$, $[BC]$ in P_1 , P_0 , P_2 respectively.

Then, by Prop. 1 (??) A , G and P_0 are aligned, as well as P_0 , H and B .

For a like reason AP_1 and H are aligned, BP_2 and G are aligned, as also are C P_1 and G , C P_2 and H .

Let (AP_0) meet the semicircle on (β) at I , and let (BP_0) meet the semicircle on (γ) in J . Join (CI) , (CJ) meeting (AP_1) , (BP_2) in K , L , and let (GL) , (HM) produced meet (AB) in F , E respectively.

Now, in the triangle AGB , the perpendiculars from A , C on the opposite sides meet in K . Therefore by the properties of triangles, (GF) is perpendicular to (AC) . Similarly (HE) is perpendicular to (BC) .

The polygon $EFGH$ is a rectangle. Let's show that it is a square.

1. It is easy to show that the lines (CJ) and (AP_0) are parallel as are the lines (BP_0) and (CI) .

$$\text{Therefore } \frac{AC}{BC} = \frac{AL}{LH} = \frac{AF}{FE}, \text{ and } \frac{BC}{AB} = \frac{CM}{MG} = \frac{BE}{EF}.$$

Hence $\frac{AF}{EF} = \frac{EF}{BE}$ (1). (All this is used in proposition 6 of the book of lemmas.)

2. Now we can notice that the right triangles AGF and EBH are similar. Thus

$$\frac{GF}{AF} = \frac{BE}{HE} = \frac{BE}{GF} \text{ (2).}$$

3. With (1) and (2) $GF^2 = EF^2$ so $GF = EF$. The rectangle $EFGH$ is a square.

Now suppose that B divides $[AC]$ such as :

$$k = \frac{AC}{CB} \text{ then } AF = kEF \text{ and } EF = kBE$$

We have

$$AB = AF + FE + EB \text{ either } AC + CB = AF + FE + EB$$

or

$$AB = kFE + FE + EF/k = \frac{k^2 + k + 1}{k} EF$$

Finally

$$\frac{AB}{GH} = \frac{k^2 + k + 1}{k}$$

2.3 Points concyclic

Lemma: points concyclic

Let P and Q be the midpoints of the semicircles (γ) and (β) respectively. Let O be the midpoint of the semicircle (α) on the opposite side of the arbelos. If the incircle of the arbelos is tangent to the semicircles (γ) and (β) and (α) at P_2, P_1, P_0 respectively, then

- (i) A, C, P_2, P_0 lie on a circle, center Q ;
- (ii) B, C, P_1, P_0 lie on a circle, center P ;
- (iii) The points A, B, P_2 and P_1 lie on a circle, center O .

Proof: [Peter Y. Woo]

- (i) Let us assume that the arbelos and the Pappus circle are constructed. Let D be the intersection of the semicircle (α) with the line perpendicular to (AB) at C . See (Figure ??).

Note that $AB \cdot AC = AD^2$ ($[DC]$ height of the right triangle ADB in D).

Consider the inversion i_A with respect to the circle $C(A, D)$ of center A that goes through D .

$i_A(B) = C$, and i_A leaves the line AB invariant. The semicircles (β) and (α) have as respective inverses the half-straight lines ℓ' and ℓ perpendicular to (AB) at C and B respectively. The semicircle (γ) is also invariant by i_A because its center belongs to the line (AB) . The incircle (δ) or $(P_0P_1P_2)$ of the arbelos is inverted into a circle tangent to the semicircle (γ) , and the lines ℓ, ℓ' at Q, P_1, P'_0 respectively.

Since the semicircle (γ) is invariant, the points A, P_2 , and Q are aligned because the image of Q is P_2 .

$CBP_1P'_0$ is a square and Q is its center thus B, Q and P'_0 are aligned.

Their inverses are therefore on a circle through A . A, C, P_2 and P_0 are cocyclic.

Let O be the midpoint of the arc \widehat{AB} . O, B, Q, P'_0 are aligned. The tangent at A to the circle $C(A, D)$ is parallel to the line (BL) so the diameter coming from A is perpendicular to the line (BL) . The center of the circle $C(A, D)$

belongs to the line (AL) and to the perpendicular bisector of $[AC]$ so it is the point P .

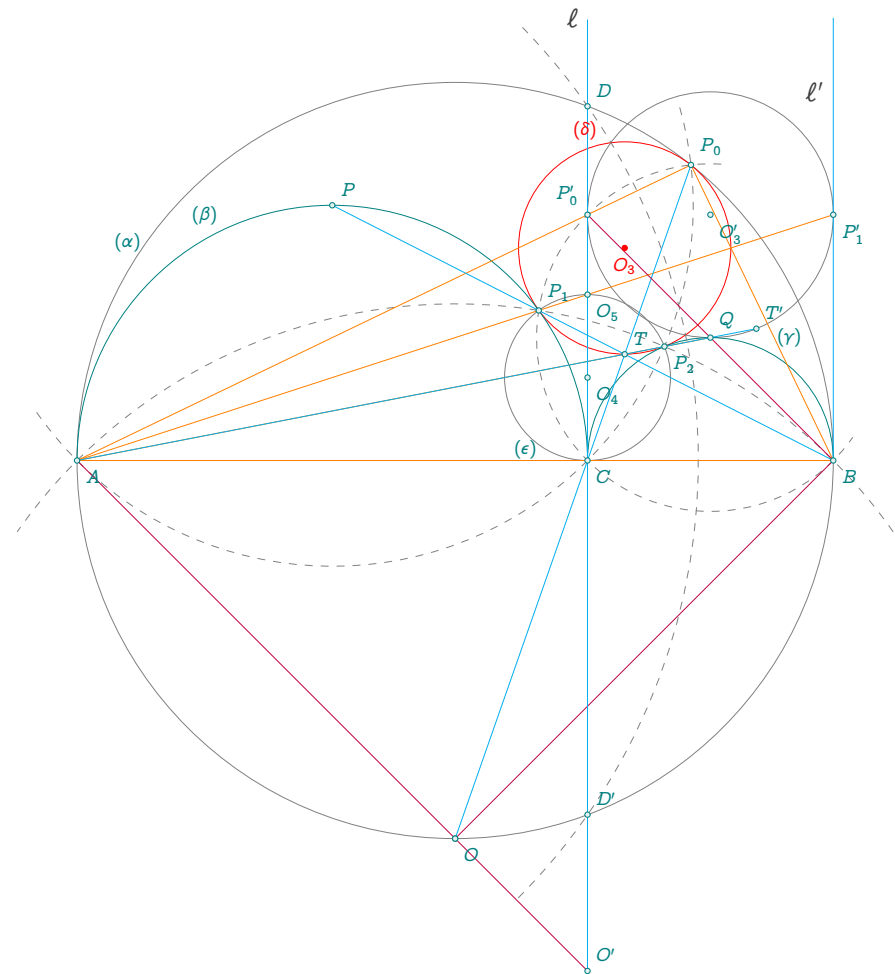


Figure 20: Points Cocyclic

- (ii) The same reasoning shows that B, C, P_1, P_0 lie on the circle $C(Q, B)$ with the inversion i_B center B through D .

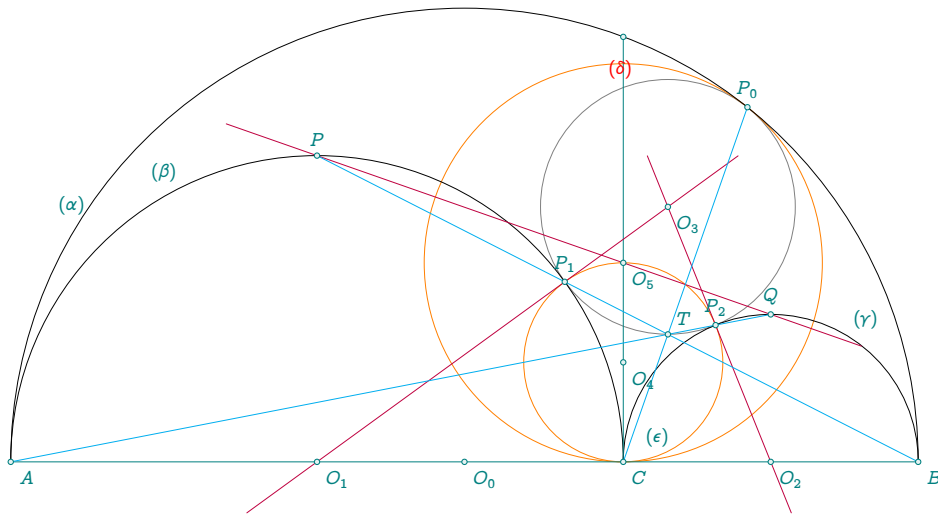


Figure 23: Pappus circle with midpoints

Some properties:

1. The point D is the intersection of the line perpendicular in C to (AB) with the circle (α) ;
2. The line (PQ) intersects (CD) at O_5 which is the north pole of the Bankoff circle;
3. O_4 is the middle of $[CO_5]$; The circle of diameter $[CO_5]$ is the Bankoff circle which intersects the circles (β) and (γ) at P_1 and P_2 ;
4. Lines (BP) and (AQ) intersect at T south pole of the circle of Pappus;
5. Line (CT) intersects the circle (α) at P_0 .

3 Bankoff Circle and other remarkable circles

3.0.1 Existence of the Bankoff circle

Let's assume an arbelos $((\alpha), (\beta), (\gamma))$ with the first pappus circle (δ) . P_0, P_1 and P_2 the points of tangency. O_0, O_1, O_2 and O_3 the centers os the semicircles and circles.

Theorem 4.3 (Bankoff)

The points of tangency of the incircle of the arbelos with the semicircles (β) and (γ) , together with (α) , are the points of tangency of the incircle (ϵ) of triangle $O_1O_3O_2$ with the sides of the triangle.

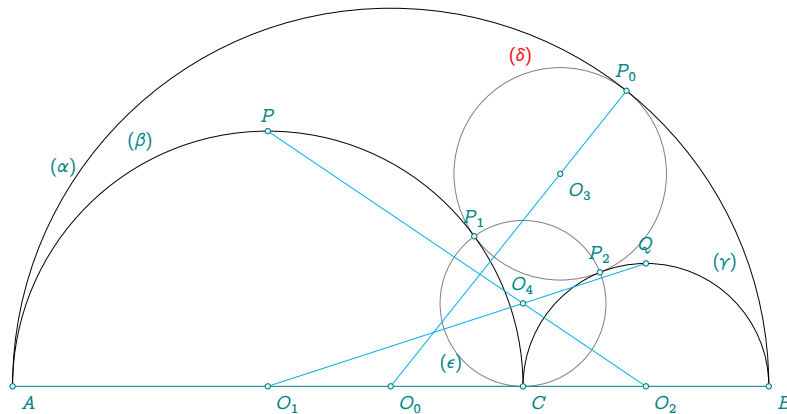


Figure 24: Bankoff Incircle

Proof and onstruction of the figure ??:

Since $O_1P_1 = O_1C$, $O_2P_2 = O_2C$, and $O_3P_2 = O_3P_1$, the points C, P_1, P_2 are the points of tangency of the incircle of triangle $O_1O_2O_3$ with its sides. The semi-perimeter p of the triangle is

$$p = a + b + r = a + b + \frac{ab}{a^2 + ab + b^2} = \frac{(a + b)^3}{a^2 + ab + b^2}$$

Its area, by application of the Heron relation, is

$$A = \sqrt{\frac{(a + b)^3}{a^2 + ab + b^2} \cdot \frac{ab}{a^2 + ab + b^2} \cdot a \cdot b} = \frac{ab(a + b)^2}{a^2 + ab + b^2}$$

The area of a triangle is also equal to the product of its half-perimeter by the radius of the inscribed circle. Thus we have

$$A = \rho p = \rho \frac{(a + b)^3}{a^2 + ab + b^2} \text{ then } \rho = \frac{ab}{a + b}$$

where ρ is the radius of the circle inscribed on the triangle $O_1O_2O_3$.

Let P and Q be the midpoints of the semicircles (AC) and (CB) respectively.

- (i) The lines O_1Q and O_2P to intersect at O_4 . The point of intersection O_4 of the lines O_1Q and O_2P verify $CO_4 = \frac{ab}{a + b}$.
- (ii) The circle with center O_4 , passing through C to intersect the semicircle (AC) at P_1 and (CB) at P_2 ,
- (iii) The lines O_1P_1 and O_2P_2 to intersect at O_3 .

The circle with center O_3 passing through P_2 touches the semicircle (β) at P_1 and also the semicircle (α) at P_0 .

Another proof:

Let's show that (P, P_1) intersects (ϵ) in B' and (PC) intersects (ϵ) in A' .

$$\widehat{P_1PC} = \frac{1}{2}\widehat{P_1O_1C} = \widehat{P_1O_1O_4} \text{ and } \widehat{P_1B'C} = \frac{1}{2}\widehat{P_1O_4C} = \widehat{P_1O_4O_1}$$

$$\widehat{P_1PC} + \widehat{P_1B'C} = \widehat{P_1O_1O_4} + \widehat{P_1O_4O_1} = 90^\circ$$

$\widehat{A'CB'} = 90^\circ$ so $[A'B']$ is a diameter of ϵ .

$\widehat{PCP'} = 90^\circ$ so P', C, B' are aligned.

A' is the orthocenter of the triangle $PP'B'$ so A' belongs to the line $P'P_1$

Moreover $(A'B')$ is parallel to (AB) . $\widehat{CA'O_4} = \widehat{A'CO_1} = 45^\circ$ as base angles of rectangular isosceles triangles.

Remark:

- From Gaspard Monge's "The Three Chords Theorem". the interior tangents common to $(\delta), (\beta), (\gamma)$ in P_2, P_1 and C concur in O_4
- (ϵ) is therefore orthogonal to (β) and (γ) .

$$A = \sqrt{\frac{(a+b)^3}{a^2+ab+b^2} \cdot \frac{ab(a+b)}{a^2+ab+b^2} \cdot a \cdot b} = \frac{ab(a+b)^2}{a^2+ab+b^2}.$$

The area of a triangle is also equal to the product of its half-perimeter by the radius of the inscribed circle. Thus we have

$$A = \rho \frac{(a+b)^3}{a^2+ab+b^2}$$

where ρ is the radius of the circle inscribed in the triangle $O_1O_3O_2$. Hence $\rho = \frac{ab}{a+b}$.

3.2 Sixth circle

Sixth Circle

The common tangent to the circles (β) and (γ) at C intersects the arc \widehat{BA} at D . Let O_5 be the point of the segment $[CD]$ such that $[CO_5]$ is a diameter of the circle (ϵ) , i.e. such that $CO_5 = 2\rho = \frac{2ab}{a+b}$. The circle (κ) with center O_5 through C is tangent to (α) at P_0 . In a golden arbelos, $\rho = a - b = \frac{b}{\varphi}$.

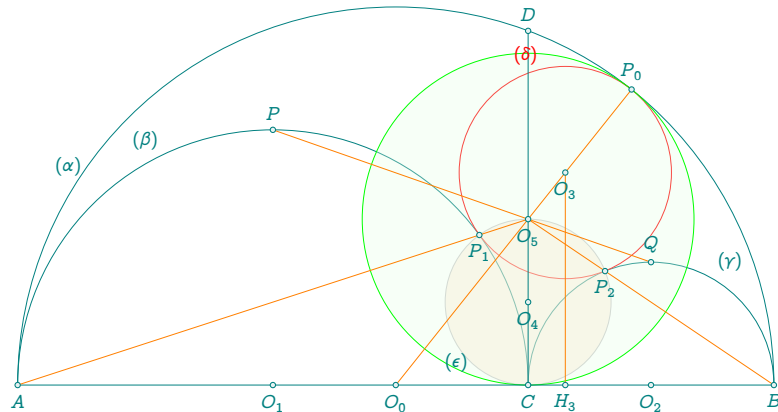


Figure 27: Sixth circle

Proof:

Let H_3 be the orthogonal project of O_3 onto (AB) .

$$O_0H_3 = \frac{-2ar}{b} + b + a - r \text{ and } r = \frac{ab(a+b)}{a^2+ab+b^2}.$$

$$O_0H_3 = \frac{2a^2b(a+b)}{b(a^2+ab+b^2)} - b - a + \frac{ab(a+b)}{a^2+ab+b^2}$$

$$O_0H_3 = \frac{2a^3b + 2a^2b^2 - (a+b)^3}{a^2+ab+b^2}$$

$$O_0H_3 = \frac{2a^3b + 2a^2b^2 - a^3 - 3a^2b - 3ab^2 - b^3}{a^2+ab+b^2}$$

$$O_0H_3 = \frac{(a-b)(a+b)^2}{(a^2+ab+b^2)}$$

$$\text{We deduce } O_0H_3^2 = \frac{(a-b)^2(a+b)^4}{(a^2+ab+b^2)^2}$$

In the right-angled triangle $O_0O_3H_3$:

$$O_3H_3^2 = O_0O_3^2 - H_3O_3^2$$

$$O_3H_3^2 = (a+b-r)^2 - \frac{(a-b)^2(a+b)^4}{(a^2+ab+b^2)^2} = \frac{4a^2b^2(a+b)^2}{(a^2+ab+b^2)^2}$$

$$\text{So } O_3H_3 = \frac{2ab(a+b)}{(a^2+ab+b^2)}$$

We finally obtain:

$$\frac{CO_5}{CO_0} = \frac{(ab)}{(a+b)(a-b)} = \frac{2ab(a+b)}{(a-b)(a+b)^2} = \frac{2ab}{(a-b)(a+b)} = \frac{O_3H_3}{O_0H_3}$$

Thus the points O_0, O_5, O_3 and P_0 are aligned.

Moreover in the right triangle O_0CO_5 we obtain $O_0O_5 = \frac{a^2+b^2}{a+b}$ which leads to

$$O_5P_0 = a+b - \frac{a^2+b^2}{a+b} = \frac{2ab}{a+b} = 2r. \quad O_5P_0 = O_5C \text{ and the circle } (\kappa) \text{ is tangent}$$

to the circle (α) as well as to the circle (δ) at point P_0 .

It's easy to show that O_5 belongs to (PQ) .

Let's show that O_5 is on (AP_1) and on (BP_2) . O_5 is on (B, M) if you consider the right triangles BMO_2 and BO_5C .

3.2.1 Seventh circle

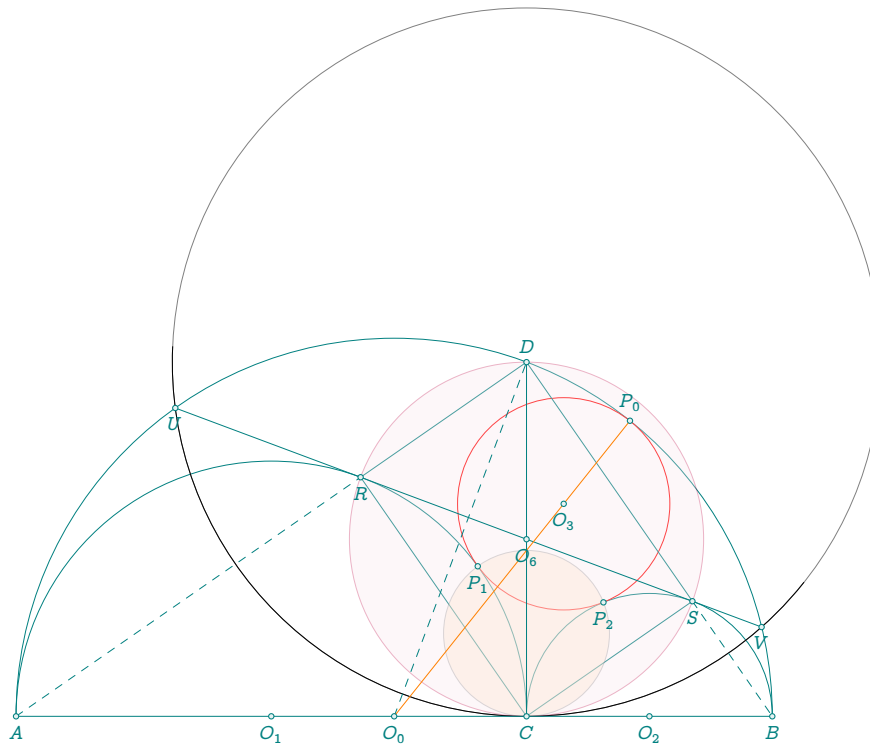


Figure 28: Seventh Circle

(AD) and (BD) intersect (β) and (γ) at R and S respectively.

The polygon $CRDS$ is a rectangle (obvious) and the middle of $[CD]$ named O_6 is the center of a circle circumscribing the rectangle.

Let us show that (RS) is a common tangent to the circles (β) and (γ) .

Consider the inversion of I_D and power DC . The circles (β) and (γ) are globally invariant because they are orthogonal at C . $I_D(A) = R$ and $I_D(B) = S$. The line (RS) is the image by I_D of the circle (α) tangent at A and B to the circles (β) and (γ) . Thus (RS) is tangent to the circles (β) and (γ) at R and S .

The line (RS) intersects the circle (α) at U and V . U and V belong to the circle (α) and to its image, so they are invariant; hence they belong to the circle with

center D passing through C .

It is easy to show that (O_0D) is orthogonal to (UV) which proves that D is the middle of the arc \widehat{UV} .

4 Midcircles

Definition

From Eric Danneels and Floor van Lamoen: A midcircle of two given circles is a circle that swaps the two given circles by inversion. Midcircles are in the same pencil of circles as the given circles. The center of the midcircle(s) is one or both of the centers of similitude. We can distinguish four cases:

- (i) The two given circles intersect: there are two midcircles with centers at the centers of similitude of the given circles;
- (ii) One given circle is in the interior of the other given circle. Then there is one midcircle with center of similitude at the internal center of similitude of the given circles;
- (iii) One given circle is in the exterior of the other given circle. Then there is one midcircle with center at the external center of similitude of the given circles. Clearly the tangency cases can be seen as limit cases of the above;
- (iv) If the circles intersect in a single point, the unique midcircle has center at the external similitude center or at internal similitude center.

Let's look at each of these cases:

- (i) If the two given circles intersect, then there are two circles of inversion through their common points, with centers at the centers of similitudes. The two midcircles bisect their angles and are orthogonal to each other. The centers of the midcircles are the internal center of similitude and the external center of similitude I and J .

Consider two intersecting circles (\mathcal{A}) and (\mathcal{B}). We can obtain the centers of similarity of these two circles by constructing \overline{EH} and \overline{FG} two diameters parallel of the circles (\mathcal{A}) and (\mathcal{B}). The line (\overline{GE}) intercepts the line (\overline{AB}) in J and the line (\overline{EF}) intercepts the line (\overline{AB}) in I . The circles (\mathcal{I}) and (\mathcal{J}) are orthogonal and are the midcircles of (\mathcal{A}) and (\mathcal{B}). The division $(A, B; I, J)$ is harmonic.

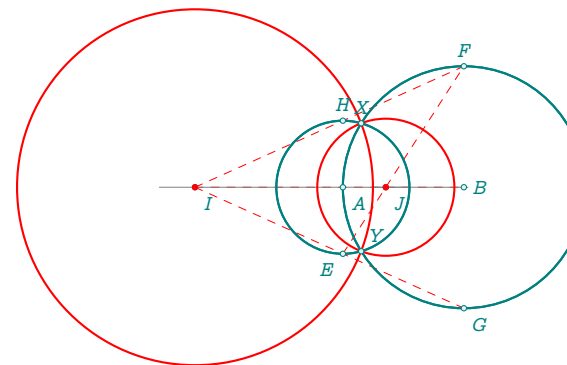


Figure 29: Given circles intersect

(ii) One given circle is in the interior of the other given circle.

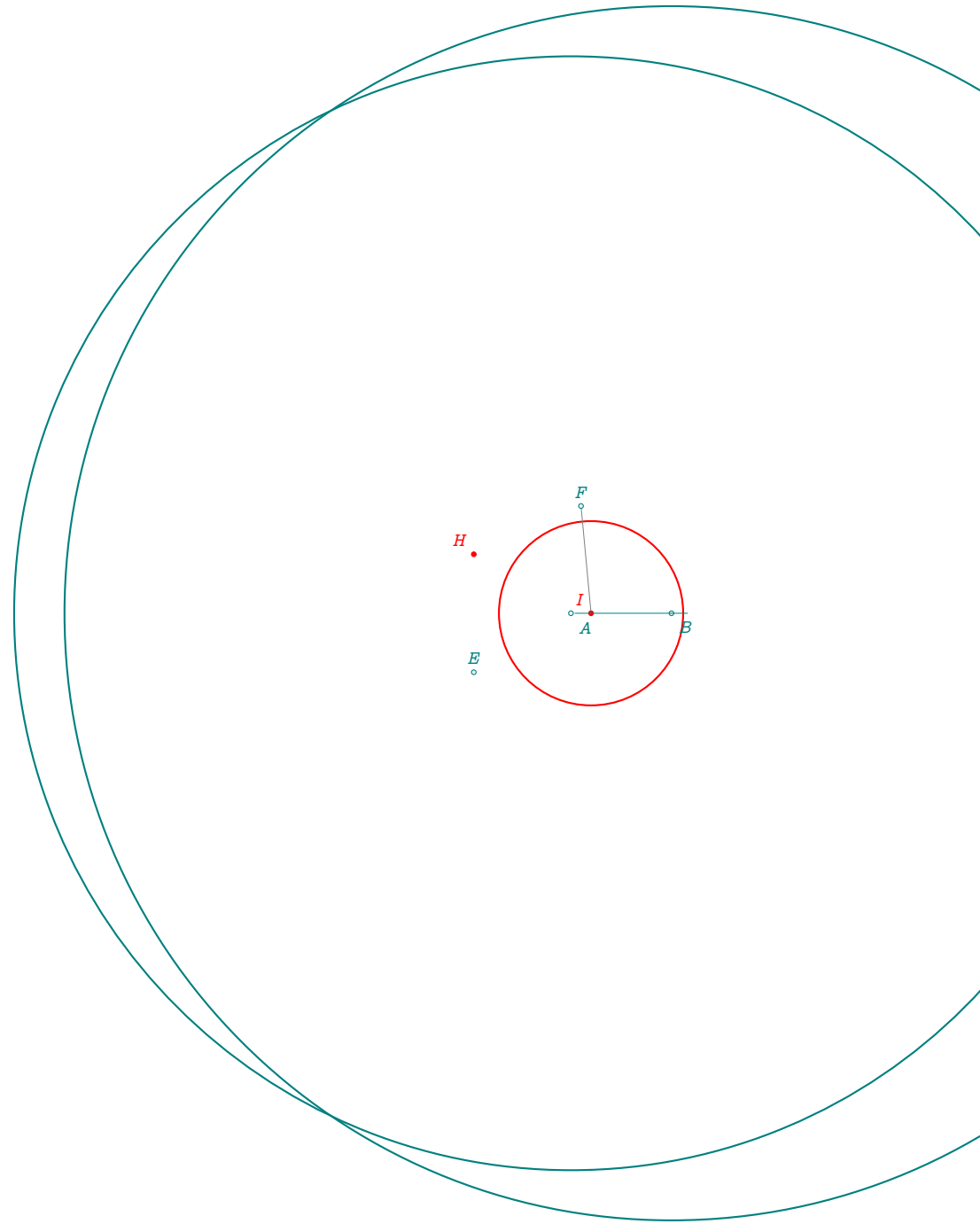


Figure 30: Given circles are external to each other

This case is a little more complicated. We'll construct the two circles (α) and (β) tangent to the two given circles. Then we construct the radical circle orthogonal to the circles (α) and (β) . Its center is the radical center as well as the center of internal similtude of circles of center A and B .

- (iii) When the two given circles are external to each other, we construct the external center of similtude of the two given circles. I is the center of external similarity of the two given circles. To obtain the inversion circle, simply note that H is such that $IH^2 = IE \times IF$
- (iv) Consider two tangent circles (\mathcal{A}) and (\mathcal{B}) ,
- (\mathcal{B}) being external and angent to (\mathcal{A}) . The construction is identical to the previous one.

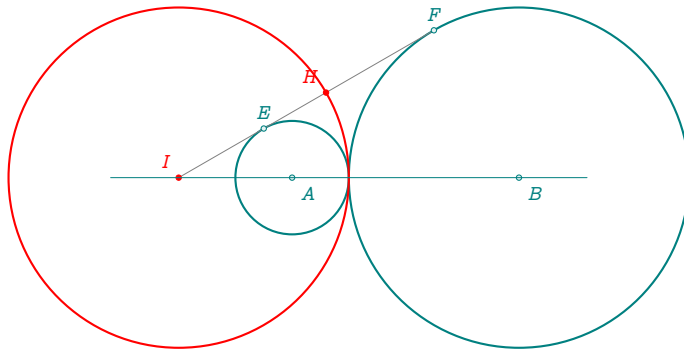


Figure 31: Fig. a

- When one of the given circles is inside and tangent to the other, the construction is easy (see figure ??).

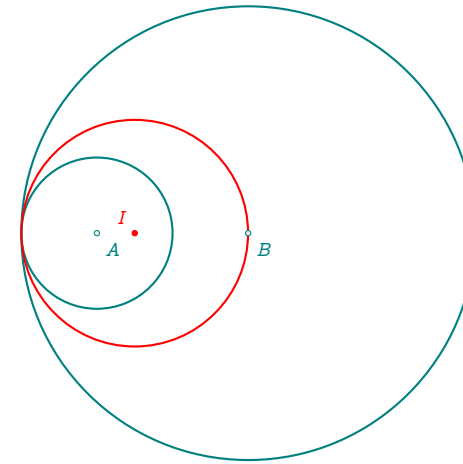


Figure 32: Fig. b

4.1 Application to the arbelos

4.1.1 The pencil of midcircles

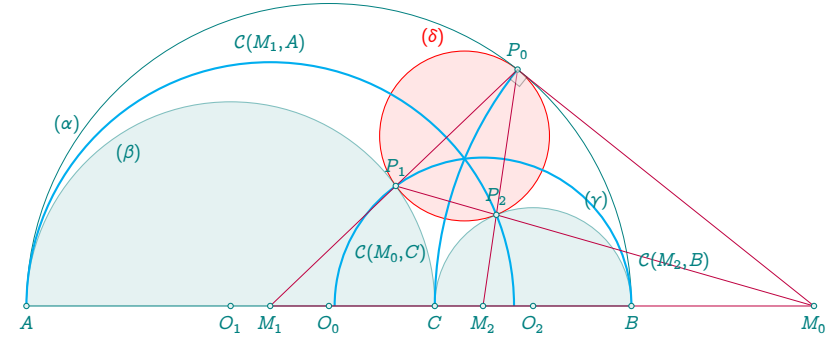
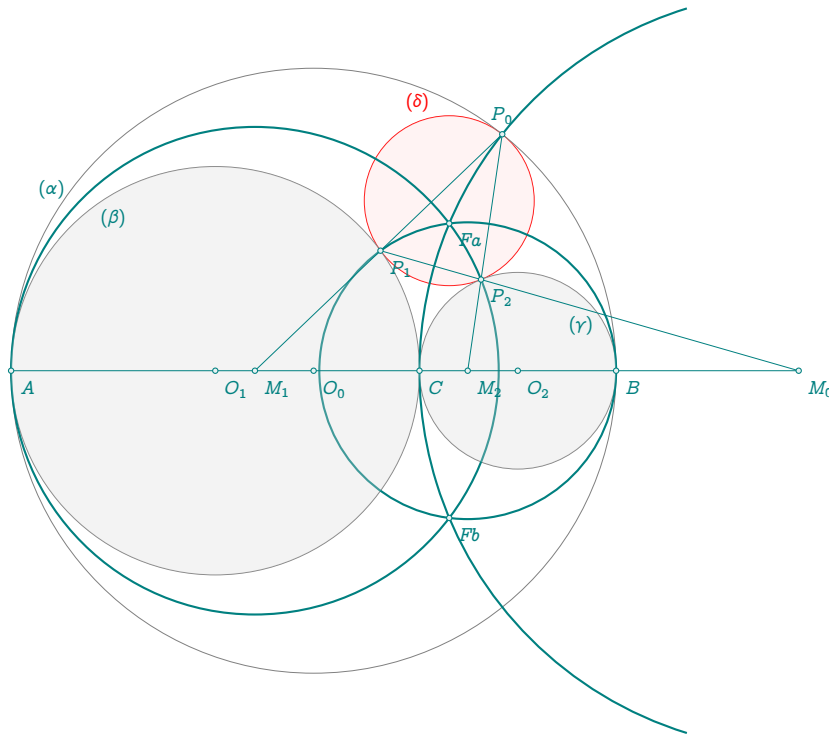


Figure 33: Midcircles

Let be M_0 , M_1 and M_2 the centers of the midcircles of (β) and (γ) , (α) and (β) and finally (α) and (γ) . These circles have M_0 , M_1 and M_2 as centers and pass through C , A and B . They can be defined as external or internal centers of similitude.

- (i) $C(M_0, C)$ this circle passes this C and P_0 .
- (ii) $C(M_1, A)$ this circle passes through A and P_2 .
- (iii) $C(M_2, B)$ this circle passes this A and P_1 .

Figure 34: Midcircles fig 1

(i) Let's call \mathcal{I}_0 the inversion of pole M_0 with respect to the circle $\mathcal{C}(M_0, C)$.

The circle (γ) has for image the circle (β) with $\mathcal{I}_0(B) = A$ and $\mathcal{I}_0(C) = C$. The Bankoff circle $(\epsilon) = \mathcal{C}(O_4, C)$ is globally invariant because it is tangent at C to the globally invariant line (AB) . The image of P_2 is a (ϵ) point and a (β) point, which is P_1 . The image of P_2 is a (ϵ) point and a (β) point, which is P_1 .

P_2 and P_1 are points of (δ) , so (δ) is also a globally invariant circle. P_0 is a point belonging to two globally invariant circles (α) and (δ) , it is therefore invariant and belongs to the inversion circle: $P_0 \in \mathcal{C}(M_0, C)$.

The first consequence is that $M_0P_0 = M_0C$. Furthermore, the line M_0P_0 is tangent to (δ) and (α) (a single point in common).

For M_0 the external center of the similitude defined by the circles (β) and (γ) can be obtained by the harmonic division $(O_1, O_2; C, M_0) = -1$.

We deduce that $M_0P_0^2 = M_0C^2 = M_0B \times M_0A$. By noticing that $M_0C = M_0B + BC = M_0B + 2b$ and $M_0A = M_0B + BA = M_0B + 2a + 2b$, it is easy to establish that

$$BM_0 = \frac{2b^2}{a-b} \text{ and } AM_0 = \frac{2a^2}{a-b}$$

Let us show that this implies that $(O_1, O_2; C, M_0)$ is a harmonic division.

The division $(O_1, O_2; C, M_0)$ will be harmonic if and only if

$$\frac{O_1M_0}{O_2M_0} = \frac{O_1C}{O_2C} \text{ or } \frac{O_0M_0}{O_1M_0} = \frac{a}{b}$$

$O_0M_0 = 2b + a + BM_0$ and $O_1M_0 = b + BM_0$ we obtain

$$\frac{2b + a + BM_0}{b + BM_0} = \frac{a}{b} \text{ and therefore } BM_0 = \frac{2b^2}{a-b}.$$

(ii) Let be M_1 the internal center of similitude of (α) and (β) . Let's call \mathcal{I}_1 the inversion of pole M_1 with respect to the circle $\mathcal{C}(M_1, A)$.

We have $\mathcal{I}_1(A) = A$ and $\mathcal{I}_1(C) = B$. Let's show that $\mathcal{I}_1(P_1) = P_0$.

The circles (β) and (δ) are globally invariant by \mathcal{I}_1 (tangent to the circles (α) and (β)).

The inversion by \mathcal{I}_1 of P_1 is a point of (α) and (δ) , so we deduce that $\mathcal{I}_1(P_1) = P_0$. The points M_1, P_1 and P_0 are aligned.

Moreover, the image of P_2 is a point on both globally invariant circles, so it is invariant and $\mathcal{I}_1(P_2) = P_2$. Thus P_2 is a point of $\mathcal{C}(M_1, A)$.

Consider the case of M_1 taking A as origin, then, the definition of the center of internal similarity:

$$AM_1 = \frac{AO_0 \times AO_1 + AO_1 \times AO_0}{AO_0 + AO_1} = \frac{2a(a+b)}{2a+b}$$

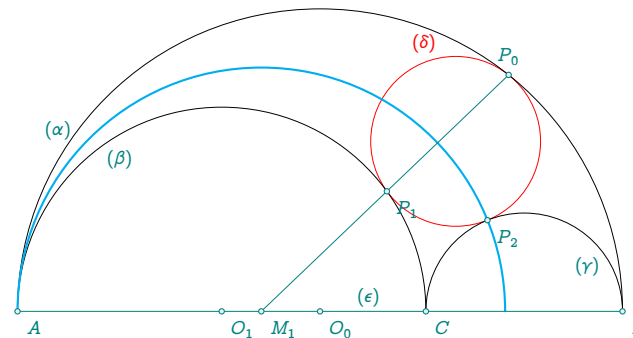


Figure 35: Midcircles fig 2

We can obtain the points M_1 using the following harmonic division $(O_0, O_1; M_1, A) = -1$.

The harmonic division $(O_0, O_1; M_1, A) = -1$ results in

$$\frac{M_1O_0}{M_1O_1} = \frac{AO_0}{AO_1} = \frac{a+b}{a} \text{ or } \frac{M_1O_0}{M_1O_1} = \frac{AO_0 - AM_1}{AM_1 - AO_1} = \frac{a+b - AM_1}{AM_1 - a} \text{ which implies}$$

$$AM_1 = \frac{2a(a+b)}{2a+b}.$$

(iii) Let's call \mathcal{I}_2 the inversion of pole M_2 with respect to the circle $\mathcal{C}(M_2, B)$. Consider the case of M_2 taking A as origin, then, the definition of the center of internal similarity:

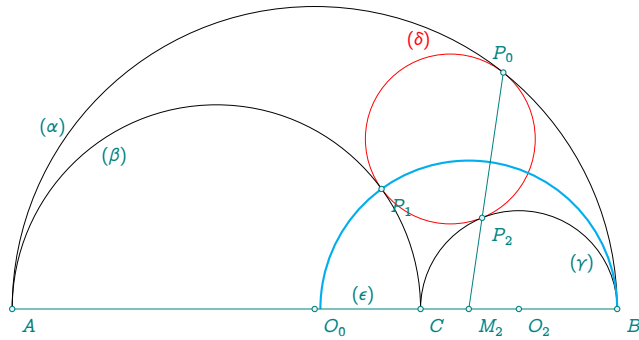


Figure 36: Midcircles fig 3

Consider the case of M_2 taking A as origin, then

$$AM_2 = \frac{bAO_0 + (a+b)AO_2}{a+2b} = \frac{b(a+b) + (a+b)(2a+b)}{a+2b}$$

$$AM_2 = \frac{2a^2 + 4ab + 2b^2}{a+2b} = \frac{2(a+b)^2}{a+2b}$$

$$AM_2 = \frac{2(a+b)^2}{a+2b}$$

We can obtain the points M_2 using the following harmonic division $(O_0, O_2; M_2, B) = -1$.

In the same way as above, we can show that the points M_2, P_2 and P_0 are aligned. Also that P_1 is invariant and a point on the circle $\mathcal{C}(M_2, B)$.

4.1.2 Ayme pappus

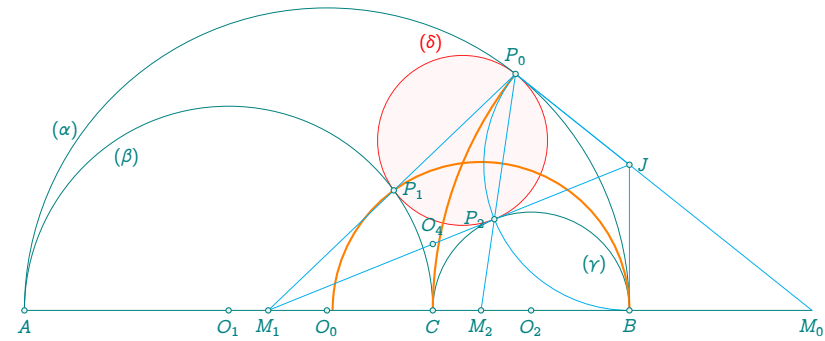


Figure 37: Pappus by Ayme

Remark: Let T_B be the tangent at B to the circles (α) and (γ) , J the point of intersection of T_B and (M_1P_2) . P_0 is the point of contact of (α) with (δ) . T the tangent to (α) and (δ) at P_0 .

According to Gaspard Monge "The three Chords theorem" T passes through J

5 Golden

5.1 $H_0 = M_2$

H_0 ? defined by $(A, B; H_0, M_0) = -1$ is such that P_0H_0 is orthogonal to (A, B) . Indeed (P_0B) is the internal bisector of $H_0P_0M_0$ and (AP_0) is the external bisector. The line (PM) is tangent to the circle at P . If H_0 is defined by $(A, B; H_0, M_0) = -1$ then the line P_0H_0 is orthogonal to the line (AB) .

If, moreover, M_0 defined by $(O_1, O_2; C, M_0) = -1$ is the external center of similarity of the circles of center O_1 and O_2 passing through C . We proved that the tangent at P_0 passes through M_0 and then the line P_0H_0 is orthogonal to the line (AB) .

$(O_0, O_2; M_2, B) = -1$ defines the center of internal similarity of the circles of center O_0 and O_2 passing through B . We know that M_2, P_2 and P_0 are aligned.

Let us show that in the case of the **golden arbelos** H_0 and M_2 are identical.

The harmonic division $(A, B; H_0, M_0) = -1$ can be translated by:
 $O_0B^2 = O_0H_0 \times O_0M_0$ sachant que O_0 est le milieu de $[AB]$.

It follows that $O_0H_0 = \frac{(a+b)^2}{3a+b}$

The harmonic division $(O_0, O_2; M_2, B) = -1$ can be translated by:

$$\frac{2}{O_0O_2} = \frac{1}{O_0M_2} + \frac{1}{O_0B}$$

Either $\frac{1}{O_0M_2} = \frac{2}{a} - \frac{1}{a+b}$ and therefore $O_0M_2 = \frac{a(a+b)}{a+2b}$.

It remains to prove that in the case of **golden arbelos**, we have $\frac{(a+b)^2}{3a+b} =$

$$\frac{a(a+b)}{a+2b}.$$

A first simplification gives $\frac{a+b}{3a+b} = \frac{a}{a+2b}$

This can be written: $(a+b)(a+2b) = a(3a+b)$ soit $a^2 + 3ab + 2b^2 = 3a^2 + ab$, which gives

$a^2 = ab + b^2$ or even $\frac{a}{b} = \frac{a+b}{a}$ which is true when $\frac{a}{b} = \varphi$.

Remark: the rectangle $O_0GJ B$ is a gold rectangle.

5.2 Particularities of the golden arbelos

Question

Let $[AB]$, the diameter of a semicircle, be divided at C so that $AB/AC = k = \varphi$. Suppose a circle drawn touching the all three semicircles. If $[GH]$ be the diameter of this circle, then show that $AB = 2\varphi GH$, $GH = a$, $E = O_2$ and $F = O_0$.

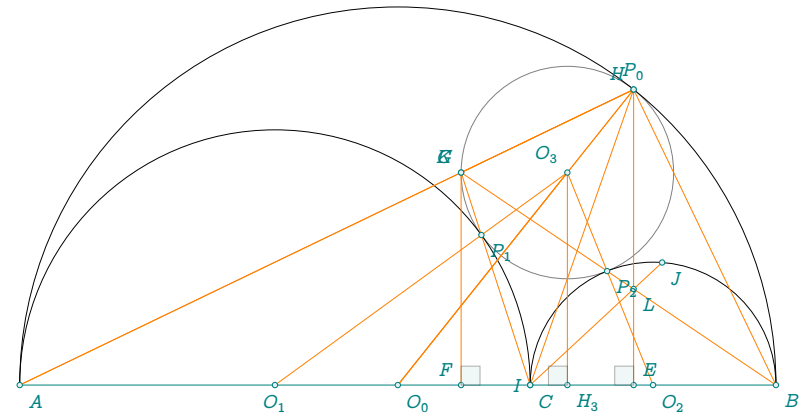


Figure 39: Pappus circle diameter

Proof:

Let $\varphi = \frac{a}{b}$ the ratio between AB and GH was given here ??

$$\frac{AB}{GH} = \frac{k^2 + k + 1}{k}$$

or, if $\varphi = \frac{a}{b}$

$$\frac{AB}{GH} = \frac{\varphi^2 + \varphi + 1}{\varphi} = \varphi + 1 + \frac{1}{\varphi} = 2\varphi$$

The problem here is to find the ratios often equal to the golden ratio.

$AP_0/AI = \varphi$

Therefore

$$\frac{AC}{CB} = \frac{AK}{KH} = \frac{AF}{EF} \quad \text{and} \quad \frac{BC}{CA} = \frac{BM}{MG} = \frac{BE}{EF}$$

hence

$$\frac{AF}{EF} = \frac{EF}{BE} \quad \text{so} \quad EF^2 = AF \times BE$$

$$AB = AF + FE + BE = \varphi^2 BE + \varphi BE + BE = (\varphi^2 + \varphi + 1)BE = 2(\varphi + 1)BE$$

$$2(a + b) = 2(\varphi + 1)BE$$

$$b(\varphi + 1) = (\varphi + 1)BE$$

so $BE = b$ and $E = O_2$

Let us show that $F = O_0$

$$\frac{AO_2}{AO_0} = \frac{2a + b}{a + b} = 1 + \frac{a}{a + b} = 1 + \frac{1}{\varphi} = \varphi$$

$$AO_2 = \varphi AO_0 \quad \text{either} \quad AE = AF + FE = \left(1 + \frac{1}{\varphi}\right)AF = \varphi AO_0$$

Finally $\varphi AF = \varphi AO_0$ and therefore $F = O_0$

$$GH = EF = O_0O_2 = a.$$

Some consequences:

1. $O_0O_2O_4$ is an isosceles triangle with $O_0O_4 = O_2O_4 = \frac{a\sqrt{5}}{2}$. This implies

$$\text{that } O_0P_0 = \frac{a\sqrt{5}}{2} + \frac{a}{2} = a\varphi.$$

$$2. \quad O_2P_2 = O_2O_4 - O_4P_2 = \frac{a\sqrt{5}}{2} - \frac{a}{2} = \frac{a}{\varphi}$$

$$3. \quad \frac{O_0B}{O_0G} = \frac{a + b}{a} = \varphi. \quad FGJB \text{ is a gold rectangle.}$$

4. In the triangle M_1JB we have $\frac{BJ}{P_2M_2} = \frac{M_1B}{M_1M_2}$. Either $BJ = \frac{M_1B \times P_2M_2}{M_1M_2}$ but in the triangle O_0GB we have $\frac{O_0G}{P_2M_2} = \frac{O_0B}{M_2B}$ which can be written $\frac{a}{P_2M_2} = \frac{a + b}{M_2B}$ or even $P_2M_2 = \frac{a \times M_2B}{a + b}$.

$$\text{Finally } BJ = \frac{M_1B \times \frac{a \times M_2B}{a + b}}{M_1M_2}.$$

We know that $M_1B = 2a$, $M_2B = \frac{2b(a + b)}{a + 2b}$ and

$$M_1M_2 = M_1B - M_2B = 2a - \frac{2b(a + b)}{a + 2b} \text{ then}$$

$$\frac{BM_2}{M_1M_2} = \frac{\varphi}{2}.$$

$$\text{We can conclude that } BJ = \frac{2a^2 \times M_2B}{(a + b)M_1M_2} = \frac{a^2 \times \varphi}{a + b} = a.$$

We knew that $BM_0 = 2a$ here ?? so it is easy to obtain the point M_0 with the compass. Then P_0 is on the circle (α) and on the circle of center BM_0 passing through C .

5.3 Pappus Radius

$EFGH$ is a square and we know $O_0 = F$ $O_2 = E$, so $EF = BG = O_0O_2 = a$

We can find the radius of the Pappus Circle with:

$$r = \frac{ab(a + b)}{a^2 + ab + b^2} = \frac{a(\varphi + 1)}{\varphi^2 + \varphi + 1} = \frac{a}{2}$$

5.4 Bankoff circle with golden arbelos

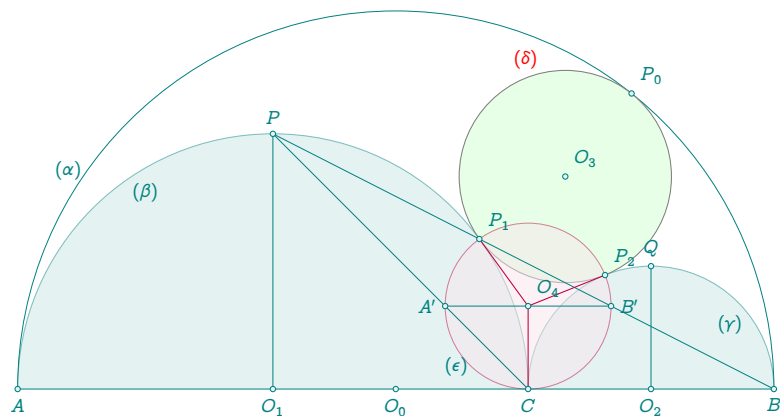


Figure 40: Golden Bankoff Circle

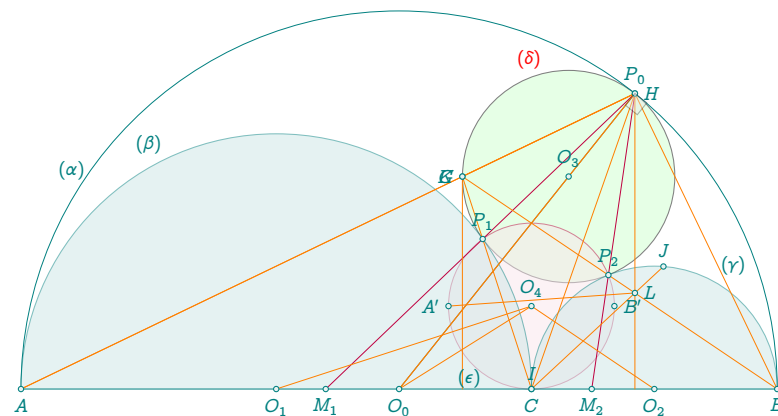
5.4.1 Bankoff Radius

$$\frac{O_5C}{PO_1} = \frac{O_2C}{O_2O_1} = \frac{O_5C}{a} = \frac{b}{a+b}$$

Let $\rho = O_5C$ we deduce from the previous equation that $\rho = \frac{ab}{a+b}$

$$\rho = \frac{ab}{a+b} = \frac{b}{1+1/\varphi} = \frac{b}{\varphi} = \frac{a}{\varphi^2} = \frac{b^2}{a}$$

5.5 Alignments



1. O_0, A', K and G are aligned.

We know that O_0, K and G are aligned, then $A'O_4 = \frac{b}{\varphi}$.

$O_0C = a - b = b\varphi - b = b(\varphi - 1) = \frac{b}{\varphi} = A'O_4$ so O_0, A', K and G are aligned.

2. A', O_4, B' and L are aligned.

L is the orthocenter of the triangle CHB .

$$\frac{O_2M}{O_0G} = \frac{O_2B}{O_0B} = \frac{b}{a+b}$$

$$\frac{O_2M}{GH} = \frac{b}{a+b}$$

$$O_2L = \frac{b}{a+b} \times \frac{2ab(a+b)}{a^2+ab+b^2} = \frac{2ab^2}{a^2+ab+b^2} = \frac{a}{\varphi^2+\varphi+1}$$

$$O_2L = \frac{2a}{\varphi^2+\varphi+1} = \frac{a}{\varphi^2} = \frac{b}{\varphi} = \rho$$

A', O_4, B' and L are aligned.

From ?? we know that O_5 the north pole of the circle (ϵ) is the center of the circle (κ)

5.6 Special segments

In the case of the **golden arbelos** $F = O_0$ with F projection of G on (AB) .

The segment $[BG]$ has length

$$BG = \sqrt{O_0G^2 + BO_0^2} = \sqrt{(a+b)^2 + a^2} = a\sqrt{\varphi^2 + 1} = a\sqrt{\varphi + 2}.$$

Remark: $BP_2 = \sqrt{2}M_2P_2$, $O_0O_4 = O_2O_4 = \frac{\sqrt{5}a}{2}$ and $O_0P_0 = \frac{\sqrt{5}a}{2} + \frac{a}{2} = \frac{a\varphi}{2}$

We have seen that $AM_2 = \frac{2(a+b)^2}{a+2b}$ which implies $BM_2 = \frac{2b(a+b)}{a+2b}$.

$$BM_2 = \frac{2b(\varphi+1)}{\varphi+2}$$

$$\frac{BM_2}{BO_2} = \frac{M_2P_2}{O_2M} \text{ which implies } BM_2 = \varphi M_2P_2$$

$$\text{so } M_2P_2 = \frac{2b(\varphi+1)}{\varphi(\varphi+2)}$$

$$\frac{BM_2}{M_1M_2} = \frac{\varphi}{2}.$$

This is because $AM_1 = 2b$, $AM_2 = \frac{2(a+b)^2}{a+2b}$ and $BM_2 = \frac{2b(a+b)}{a+2b}$.

We deduce $M_1M_2 = \frac{2(a+b)^2}{a+2b} - 2b = \frac{2(a^2+ab-b^2)}{a+2b}$ then

$$\frac{BM_2}{M_1M_2} = \frac{2b(a+b)(a+2b)}{2(a+2b)(a^2+ab-b^2)} = \frac{b(a+b)}{(a^2+ab-b^2)}$$

$$\frac{BM_2}{M_1M_2} = \frac{\varphi+1}{\varphi^2+\varphi-1} = \frac{\varphi^2}{2\varphi} = \frac{\varphi}{2}$$

Relations with centers:

$O_0C = a-b = b(\varphi-1)$, $O_2B = b$, $O_0B = a+b = b\varphi^2$, $O_0O_1 = b$ and $O_0O_2 = a = b\varphi$

Relations with centers of the midcircles (M_1, M_2, M_0):

$$AM_1 = 2b, AM_2 = \frac{2(a+b)^2}{a+2b}, M_1M_2 = \frac{2(a^2+ab-b^2)}{a+2b}$$

Relationship with the tangency points

5.7 Golden triangle in the golden Arbelos

Identifying golden and equilateral triangles that arise from the golden Arbelos

Article in The Mathematical Gazette · October 2019

Author : Aldo Scimone

Global Journal of Advanced Research on Classical and Modern Geometries

<https://www.researchgate.net/publication/336486345> A note on regular pentagons arising from the golden arbelos by HIROSHI OKUMURA

5.7.1 Golden triangle

In order to simplify the demonstrations, we will use the following notations:

$AC = 2a$ and $CB = 2b$

O_0, O_1 and P_2 being the centers of the semi-circles which constitute the arbelos:

$AO_0 = a+b$, $O_0C = a-b$, $O_1O_0 = b$, $O_1O_2 = 2a+2b-a-b = a+b$ and finally

$O_2O_0 = b+(a-b) = a$.

C dividing AB in a divine proportion we can write :

$$\frac{AC}{BC} = \frac{a}{b} = \frac{a+b}{a} = \varphi$$

We want to prove that the triangle O_0MC is golden.

To demonstrate the property we prove that $\frac{O_0M}{O_0C} = \varphi$

$$\frac{AB}{AC} = \varphi \text{ then } AC = \frac{AB}{\varphi} = \frac{2(a+b)}{\varphi}$$

$$O_0C = a-b = AC - AO_0 = \frac{2(a+b)}{\varphi} - (a+b) = \frac{2(a+b)}{\varphi} - \frac{\varphi(a+b)}{\varphi}$$

$$O_0C = \frac{2\varphi(a+b) - (a+b)}{\varphi} = \frac{(2\varphi-1)(a+b)}{\varphi} = \frac{(a+b)}{\varphi^3} \text{ (see ??)}$$

$$O_0C = \frac{a+b}{\varphi^3} = \frac{(a+b)(2-\varphi)}{\varphi}$$

$$O_1L = a = \frac{a+b}{\varphi}$$

$$O_0I = \frac{a-b}{2}$$

$$O_1I = O_1O_0 + O_0I = b + \frac{a-b}{2} = \frac{a+b}{2}$$

$$O_0M^2 = O_0I^2 + IM^2 = O_1M^2 - O_1I^2 + O_0I^2$$

$$O_0M^2 = \frac{(a+b)^2}{\varphi^2} - \frac{(a+b)^2}{4} + \frac{(a+b)^2(2-\varphi)^2}{4\varphi^2}$$

$$O_0M^2 = \frac{4(a+b)^2 - \varphi^2(a+b)^2 + (a+b)^2(2-\varphi)^2}{4\varphi^2}$$

$$O_0M^2 = \frac{(a+b)^2[4 - \varphi^2 + (2-\varphi)^2]}{4\varphi^2}$$

$$O_0M^2 = \frac{(a+b)^2(2-\varphi)}{\varphi^2} = \frac{(a+b)^2}{\varphi^4} \quad O_0L = \frac{a+b}{\varphi^2}$$

With the result $\frac{O_0M}{O_0C} = \varphi$, so the triangle O_0MC is golden.

Now let's prove that O_1MC is golden.

$$O_1L = \frac{a+b}{\varphi} \text{ and } O_0C = \frac{a+b}{\varphi^2}$$

$$\frac{O_1M}{MC} = \varphi \text{ so the triangle } O_1MC \text{ is golden too.}$$

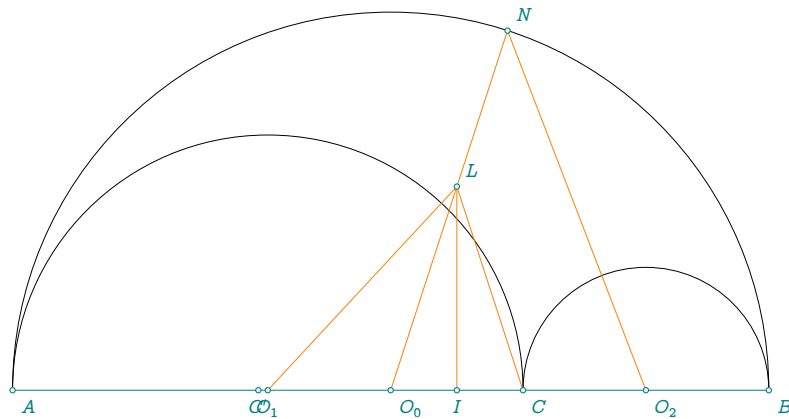
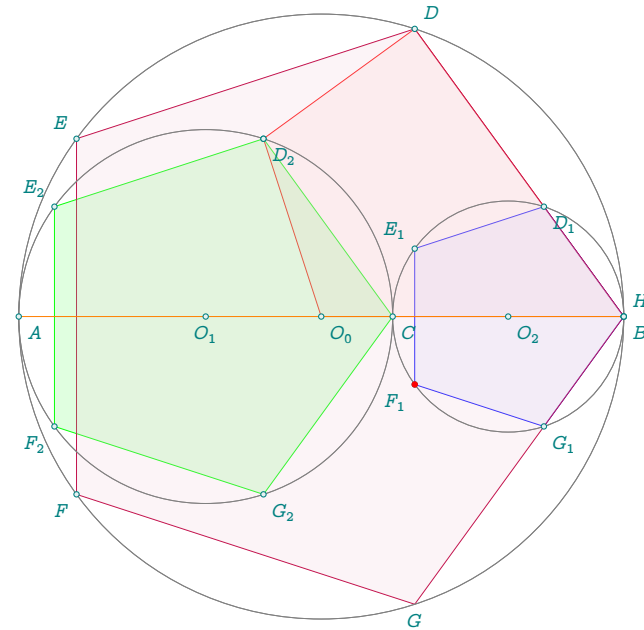
The line (O_0M) intersects the circle β at N . Let us draw the line parallel to (MC) passing through N . It cuts $[AB]$ at P . Let us show that $P = O_2$ and that the triangle O_0NO_2 is a golden triangle.

$$\text{We have } \frac{O_0N}{O_0M} = \frac{O_0P}{O_0C}. \text{ Either } (a+b) \times \frac{\varphi^2}{a+b} = \frac{O_0P \times \varphi^3}{a+b}$$

$$\text{Then } O_0P = \frac{a+b}{\varphi} = a \text{ also } P = O_2.$$

The triangle O_0NO_2 is isosceles and the angle $\widehat{O_2O_0N} = 72^\circ$ so it is a golden triangle or $\frac{O_0N}{O_0O_2} = \frac{a+b}{a} = \varphi$ and the conclusion is identical.

5.7.2 Regular pentagons in golden Arbelos



5.8 Golden Midcircles

$H_0 = M_2; E = O_2; F = O_0$

$\backslash\text{tkzDivHarmonic}[\text{int}](A,C,M_0)$ $\backslash\text{tkzGetPoint}\{M_1\}$

$\backslash\text{tkzDivHarmonic}[\text{int}](A,B,M_0)$ $\backslash\text{tkzGetPoint}\{M_2\}$

instead of

$\backslash\text{tkzDivHarmonic}[\text{int}](O_0,O_1,A)$ $\backslash\text{tkzGetPoint}\{M_1\}$

$\backslash\text{tkzDivHarmonic}[\text{int}](O_0,O_2,B)$ $\backslash\text{tkzGetPoint}\{M_2\}$

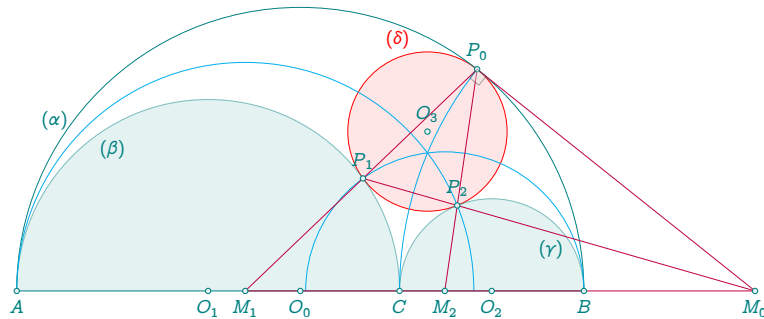


Figure 41: Golden Midcircles

We know $AM_1 = \frac{2a(a+b)}{2a+b}$.

With golden arbelos

$$AM_1 = \frac{2b\varphi(b\varphi+b)}{2b\varphi+b} = \frac{2b(\varphi^2+\varphi)}{2\varphi+1} = \frac{2b\varphi(\varphi+1)}{\varphi+\varphi^2} = 2b$$

With golden arbelos, $BM_0 = \frac{2b^2}{b\varphi-b} = \frac{2b}{\varphi-1} = 2b\varphi = 2a$

We deduce that $AM_0 = 2(a+b) + BM_0 = 2(a+b) + \frac{2b^2}{a-b} = \frac{2a^2}{a-b}$

Then $AM_0 = 4a + 2b$

On the one hand $M_0B \times M_0A = \frac{4a^2b^2}{(a-b)^2}$, on the other hand

$$M_0C^2 = (2b + M_0B)^2 = \left(2b + \frac{2b^2}{a-b}\right)^2 = \left(\frac{2b(a-b) + 2b^2}{a-b}\right)^2 = \frac{4a^2b^2}{(a-b)^2}$$

$M_0C = 2a + 2b = AB$ and $M_0C^2 = 4(a+b)^2$

5.8.1 Seventh circle in golden arbelos

It is easy to establish that $\frac{AD}{AR} = \frac{AR}{RD} = \varphi$ and $\frac{AD}{AR} = \frac{AO_1}{AO_2} = \frac{a+b}{a} = \varphi$

We know that $CD = 2\sqrt{ab}$. In the same way $\frac{BD}{SD} = \varphi$

Using similar triangles and the Pythagorean theorem, it is easy to establish that $AD^2 = 4(a^2 + ab)$ and that $BD^2 = 4(b^2 + ab)$. It follows that $\frac{AD^2}{BD^2} = \frac{a^2 + ab}{b^2 + ab}$

or even that $\frac{AD^2}{BD^2} = \frac{\varphi^2 + \varphi}{1 + \varphi} = \varphi$. We deduce that $\frac{AD}{BD} = \sqrt{\varphi}$. In the same way

$$\frac{RC}{RD} = \sqrt{\varphi}$$

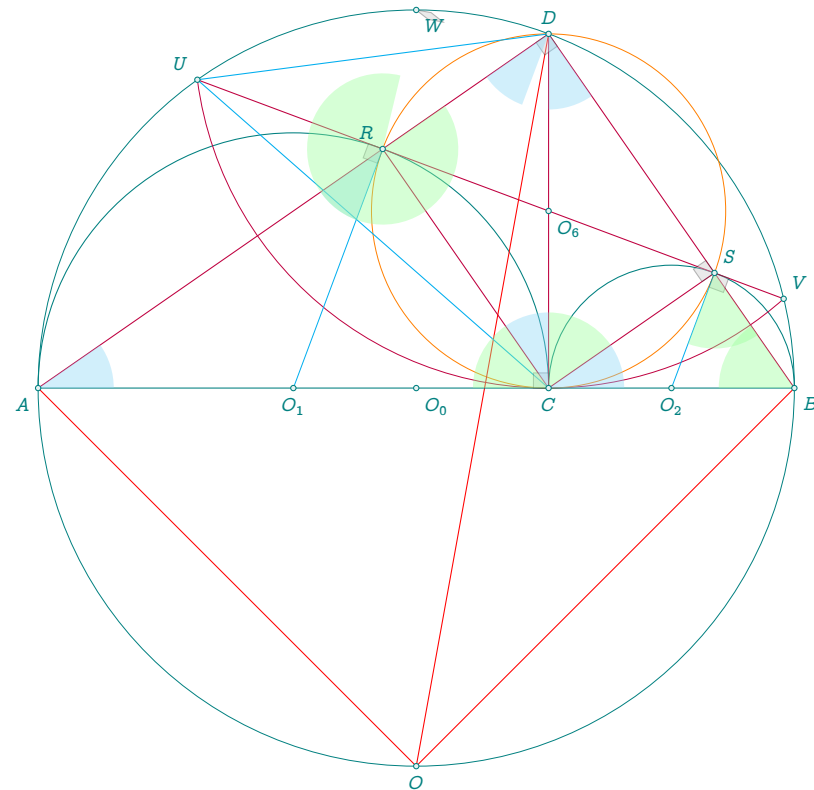


Figure 42: Seventh circles in golden arbelos

5.8.2 First consequences

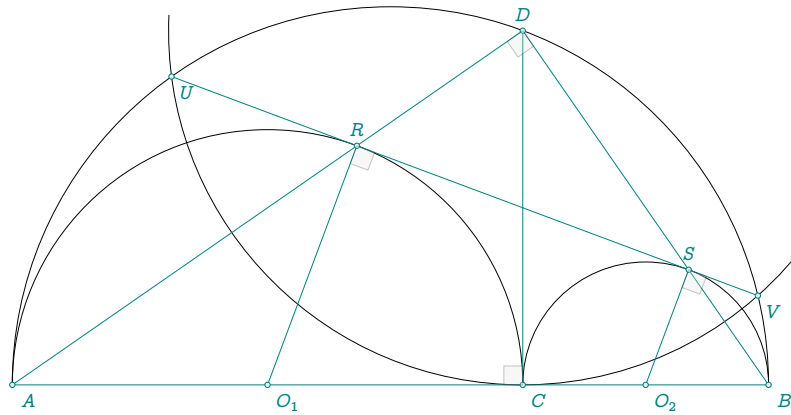


Figure 43: Consequences

It is quite easy with the help of similar right-angled triangles to establish :

$$BD = AC = 2a, CD = 2\sqrt{ab} = 2b\sqrt{\varphi} \text{ and } AD = 2a\sqrt{\varphi}$$

$$\frac{AD}{AR} = \frac{BD}{DS} = \varphi$$

6 Arbelos - References

1. Art of Problem Solving, "A First Pappus circle." High School Olympiads;
<https://artofproblemsolving.com/community/c6h619359>
2. Leon Bankoff, "A Mere Coincidence." Mathematics Newsletter, Los Angeles City College, Nov. 1954;
<http://www.retas.de/thomas/arbelos/Bankoff.pdf>
3. Christer Bergsten, "Magic Circles in the Arbelos" The Mathematics Enthusiast Volume 7;
<https://www.math.tamu.edu/~harold.boas/preprints/arbelos.pdf>
4. Alexander Bogomolny, "Arbelos - the Shoemaker's Knife"
<https://www.cut-the-knot.org/proofs/arbelos.shtml>
5. Alexander Bogomolny, "The Book of Lemmas: Proposition 1"
<https://www.cut-the-knot.org/Curriculum/Geometry/BookOfLemmas/BOL1.shtml>
6. Harold P. Boas, "Reflections on the Arbelos" MAA, 113 (March 2006);
<https://www.math.tamu.edu/~harold.boas/preprints/arbelos.pdf>
7. Eric Danneels and Floor van Lamoen, "Midcircles and the Arbelos"
<https://forumgeom.fau.edu/FG2007volume7/FG200707.pdf>
8. Eric Danneels and Floor van Lamoen, "Midcircles and the Arbelos"
<https://forumgeom.fau.edu/FG2007volume7/FG200707.pdf>
9. Baptiste Gorin, "Une étude de l'arbelos."
https://irem.univ-reunion.fr/IMG/pdf/Gorin_Arbelos.pdf
In this article you will find almost everything you need to know about the arbelos, with detailed figures and demonstrations.
10. Kurt Hofstetter, "Another 5-step Division of a Segment in the Golden Section"
<https://forumgeom.fau.edu/FG2004volume4/FG200402.pdf>
11. Dick Klingens, "Een, volgens Pappos, oud probleem: de arbelos"
<http://www.pandd.demon.nl/arbelos.htm>
12. Brian Mortimer, "The Geometry of The Arbelos" Carleton University, April, 1998.
https://schule.bayernport.com/arbelos/arbelos_06.pdf
13. Hiroshi Okumura and Masayuki Watanabe, "The Archimedean Circles of Schoch and Woo" Forum Geometricorum Volume 4 (2004) 27-34.
<https://forumgeom.fau.edu/FG2004volume4/FG200404.pdf>
14. Hiroshi Okumura and Masayuki Watanabe, "Characterizations of an infinite set of Archimedean circles" Forum Geometricorum Volume 7 (2007) 121-123.
<https://forumgeom.fau.edu/FG2007volume7/FG200716.pdf>
15. Hiroshi Okumura, "Archimedean circles passing through a special point"
<https://ijgeometry.com/wp-content/uploads/2015/03/1.pdf>

16. Peter Y. Woo, "Simple Constructions of the Incircle of an Arbelos" Forum Geometricorum, Volume 1 (2001) 133-136.
<https://forumgeom.fau.edu/FG2001volume1/FG200119.pdf>
17. Weisstein, Eric W. "Arbelos." From MathWorld—A Wolfram Web Resource.
<https://mathworld.wolfram.com/Arbelos.html>
18. Weisstein, Eric W. "Bankoff Circle." From MathWorld—A Wolfram Web Resource.
<https://mathworld.wolfram.com/BankoffCircle.html>