Golden Arbelos 1.60c
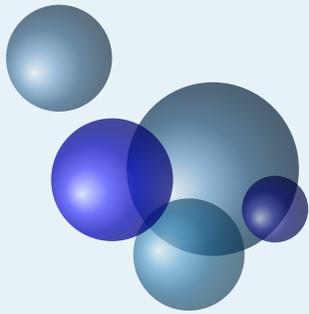
Alain Matthes

Golden Arbelos

Alain Matthes

This document presents some properties of the arbelos built around the divine proportion. Most of the demonstrations have been found on the Internet. I have quoted the authors when possible. All figures were obtained with my packages tkz-elements.

and tkz-euclide.

☞ You will find most of the examples in this document and many other examples on my website : altermundus.fr.

Please report typos or any other comments to this documentation to Alain Matthes

AlterMundus

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List of Figures

1 Arbelos – References

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2 Presentation

In this paper I will focus on the different constructions of the first Pappus circle in an arbelos, indicating which geometrical properties one can note. In the particular case of arbelos built from a divine proportion, I will indicate which special constructions can be used as well as which geometrical peculiarities result from it.

2.1 Definition
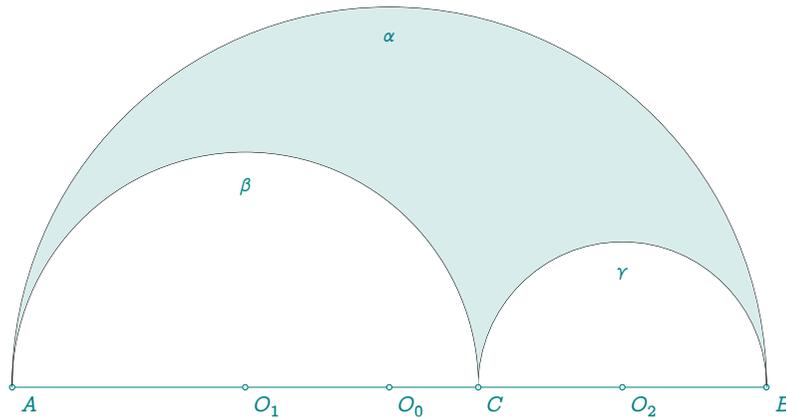
Figure 1: Arbelos

> **Arbelos Definition by Archimedes' Book of Lemmas**
>
> *If $[AB]$ be the diameter of a semicircle and C any point on $[AB]$, and if semicircles be described within the first semicircle and having $[AC]$, $[CB]$ as diameters respectively, the figure included between the circumferences of the three semicircles is called* **arbelos**. *Let β, γ, α be the semicircles of respectives diameters $[AC]$, $[CB]$ and $[AB]$, then the arbelos bounded by the three semicircles is denoted (β, γ, α) or (A, C, B).*

In the rest of the document, we will refer to a, b et $a + b$ the respective radii of the semicircles β, γ and α. O_1, O_2 and O_0 will be the respective centers of the semicircles.
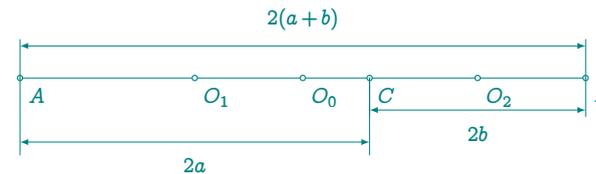
Figure 2: notations

2.2 About the Devine Proportion

2.2.1 Definition

> **Golden Ratio**
>
> *In mathematics, expressed algebraically, for quantities a and b with $a > b > 0$,*
>
> $$\frac{a+b}{a} = \frac{a}{b} = \varphi$$
>
> *where the Greek letter φ represents the* **golden ratio** *or* **devine proportion**. *In other words, two quantities are in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities. [wikipedia]*

In the package `tkz-euclide`, you can use the macro `\tkzPhi` to get the value of golden ratio: $\varphi \approx 1.618034$.
We saw above that the Golden Ratio has the property:

$$\frac{a+b}{a} = \frac{a}{b}$$

We can split the first fraction

$$\frac{a}{a} + \frac{b}{a} = \frac{a}{b}$$

Either $1 + \dfrac{1}{\varphi} = \varphi$ with $\varphi = \dfrac{a}{b}$

Also φ is a positive solution of :

$$x^2 - x - 1 = 0$$

That is to say $\varphi = \dfrac{1 + \sqrt{5}}{2}$

2.2.2 Algebraic Properties

– Powers of φ

Formula	
φ^2	$= \varphi + 1$
φ^3	$= \varphi^2 + \varphi = 2\varphi + 1$
φ^4	$= 3\varphi + 2$
φ^n	$= \varphi^{(n-1)} + \varphi^{(n-2)}$

– Powers of $\dfrac{1}{\varphi}$

Formula	
$\dfrac{1}{\varphi}$	$= \varphi - 1$
$\dfrac{1}{\varphi^2}$	$= 2 - \varphi = 1 - \dfrac{1}{\varphi}$
$\dfrac{1}{\varphi^3}$	$= 2\varphi - 3 = \dfrac{2}{\varphi} - 1$

2.2.3 Geometric Properties

– **Golden section of a line segment**

This is the simplest example of the golden ratio in relation to geometry. It's a special line segment portion of a line segment, called the golden segment, illustrated here:

Figure 3: Golden Segment

We consider the segment $[AB]$ and we decide to define a point C, situated between A and B in order to have:

$$\frac{AC}{CB} = \frac{AB}{AC}$$

The point C defines a devine proportion on the segment $[AB]$. AC is the golden section of the line segment $[AB]$.

It's easy to get C Just use the macro of `tkz-euclide`: `\tkzDefGoldenRatio(A,B)` `\tkzGetPoint{C}`.

With $AC = 2a, CB = 2b$ a>b

We transformed this relation by doing a cross product :

$$AC^2 = AB \cdot CB$$

which becomes

$$4a^2 = 2(a+b) \cdot 2b$$

That we can write:

$$a^2 = ab + b^2 \text{ or } \frac{a}{a} + \frac{b}{a} = \frac{a}{b}$$

– `lemma`: *The midpoints of the arcs of an arbelos is aligned.*

To help you find the midpoint of an arc, I created the macro `\tkzDefMidArc` so `\tkzDefMidArc(O,A,B)` gives the midpoint of the arc $\overparen{AB}$ on the circle with center O.
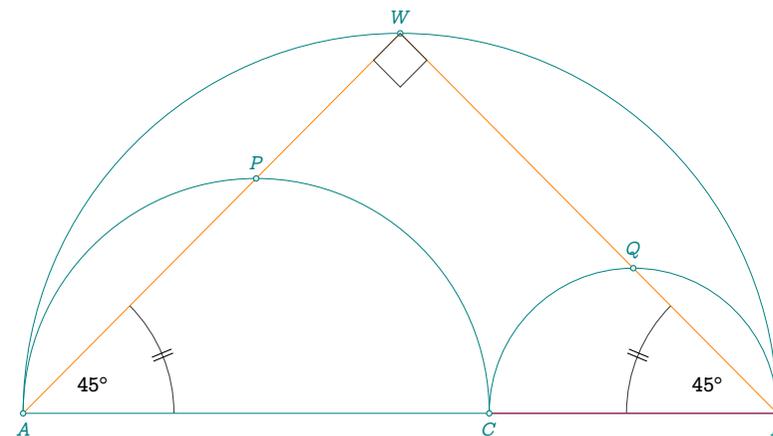
Figure 4: Midpoints of the arcs

With a **golden arbelos** see figure 4

$$\frac{AC}{BC} = \frac{AP}{PW} = \varphi \quad \text{and} \quad \frac{BQ}{QW} = \frac{BC}{AC} = \frac{1}{\varphi}$$

– Obtaining the divine proportion with a ruler and a compass.
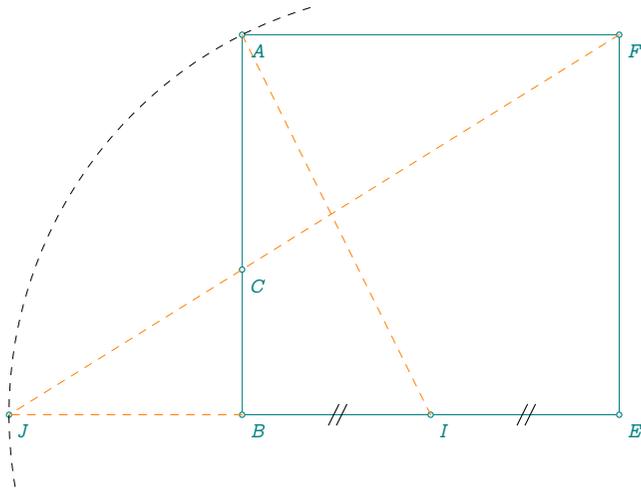
Figure 5: $[AC]$ is the golden segment of $[AB]$

1. First we construct a square $ABEF$;

2. Then we construct the midpoint I of segment $[BE]$;

3. The arc of center I passing through A intersects the half-line $[EB)$ at J;

4. Finally the line (JF) intersects the segment $[AB]$ at C.

Proof: $[AC]$ is the golden segment of $[AB]$

If $AF = AB = x$ then $AI = IJ = \sqrt{x^2 + \frac{x^2}{4}} = \frac{x\sqrt{5}}{2}$

$$BJ = IJ - IB = \frac{x\sqrt{5}}{2} - \frac{x}{2} = \frac{-1+\sqrt{5}}{2}x = \frac{x}{\varphi}$$

Finally (The triangles BCJ and ACF are similars)

$$\frac{AC}{CB} = \frac{AF}{BJ} = \varphi$$

Remark: $\dfrac{EJ}{EB} = \varphi$

– Divide a segment according to the divine proportion with a ruler and a compass.
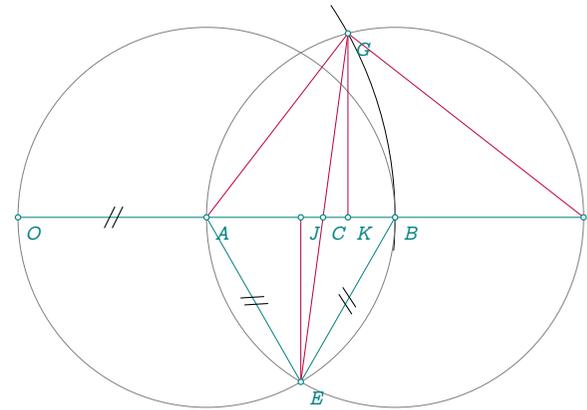
Figure 6: Divide a segment

1. Let J and K be such that $AJ = \dfrac{AB}{2}$ and $JK = \dfrac{AB}{4}$

2. $GK^2 = AK \times KI$ then $GK^2 = \dfrac{3}{4}AB \times \dfrac{5}{4}AB$ then $GK = \dfrac{\sqrt{15}}{4}AB$

3. $JE = \dfrac{\sqrt{3}}{2}AB$ so $\dfrac{JE}{GK} = \dfrac{4\sqrt{3}}{2\sqrt{15}} = \dfrac{2}{\sqrt{5}}$

4. $\dfrac{JC}{CK} = \dfrac{JE}{GK}$ and therefore $JC = \dfrac{2}{\sqrt{5}} \times (JK - JC)$. It follows that

$$JC = \frac{2}{\sqrt{5}+2}JK = \frac{(\sqrt{5}-2)}{2}AB$$

5. $AC = AJ + JC = \dfrac{AB}{2} + \dfrac{(\sqrt{5}-2)}{2}AB = \dfrac{(\sqrt{5}-1)}{2}AB = \dfrac{2}{\sqrt{5}+1)}AB$

6. Finally $\dfrac{AB}{AC} = \varphi$

Remark: It is possible to obtain the point G by drawing the circle of center O passing through B.

– **Golden Rectangle**
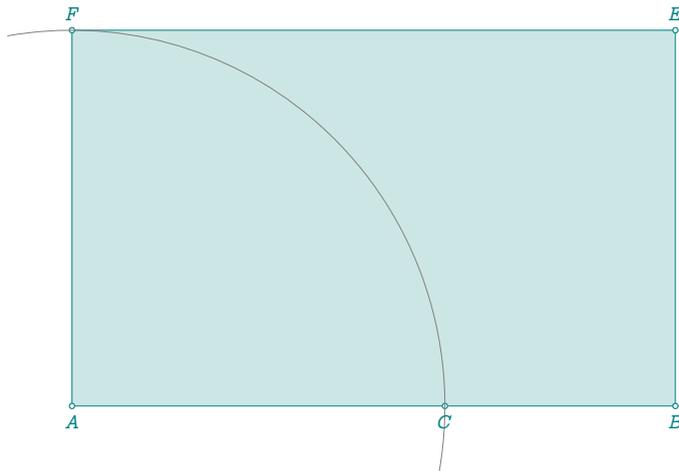
Figure 7: Golden Rectangle

Figure 8: Golden triangle

The rectangle $ABEF$ is a golden rectangle because the ratio of the length AB to the width BE is equal to the golden ratio φ. The construction is easy with **tkz-euclide** and its macro
\tkzDefGoldRectangle(A,B).
It is of course possible to use the previous construction to build this rectangle with the compass. C divides $[AB]$ with the divine proportion and then $AF = AC$

– **Golden Triangle or Sublime Triangle**

The sublime triangle (figure 8) can be obtained from the golden rectangle. It is also possible to use a macro of **tkz-euclide**:
\tkzDefTriangle[golden](B,C)
\tkzGetPoint{A}.

In the figure (8) we have $\dfrac{a}{b} = \varphi$ and $\theta = 36°$

– **Kepler'Triangle** A Kepler triangle is a special right triangle with edge lengths in geometric progression (Like ABC in the figure 9). The ratio of the progression is $\sqrt{\varphi}$, where φ is the golden ratio, and the progression can be written: $1 : \sqrt{\varphi} : \varphi$.
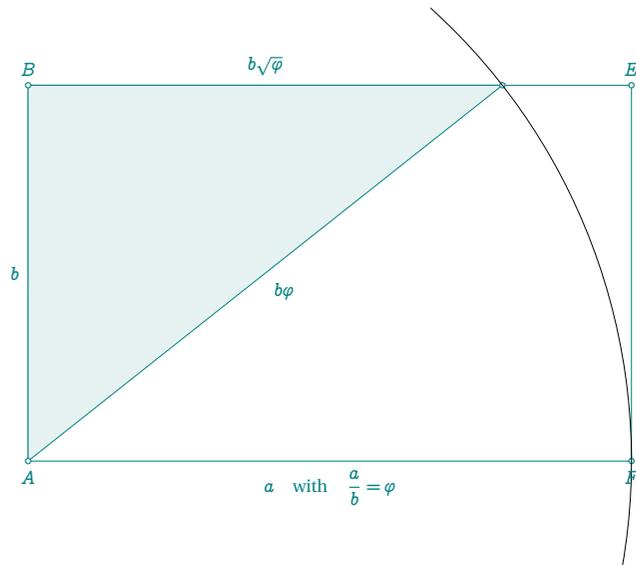
Figure 9: Kepler'Triangle

– **Right triangle and golden ration**
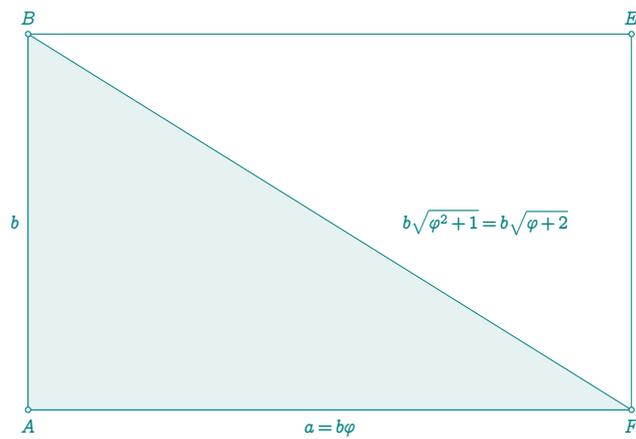
Figure 10: Right Triangle

2.3 Golden Arbelos

> **Definition**
>
> *Let β, γ, α be the semicircles with the respective radii $a, b, a+b$. If the positive numbers a and b $(a > b)$ are in a* **golden ratio** *in other words if*
>
> $$\frac{a+b}{a} = \frac{a}{b} = \varphi$$
>
> *then this special type of arbelos is called golden arbelos.*
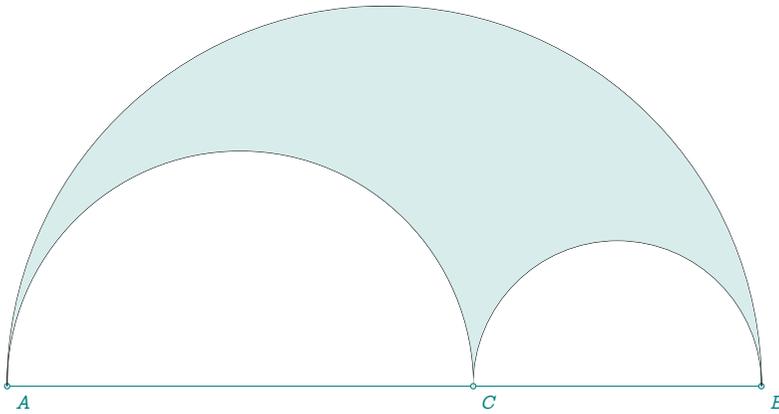
Figure 11: Golden Arbelos $\dfrac{AC}{CB} = \dfrac{AB}{AC} = \varphi$

2.3.1 Construction of the Golden Arbelos

We can consider two cases. Suppose we want to obtain the arbelos (A, C, B) then we have to examine the construction from the couple of points (A, C) or from the couple (A, B). We will use the construction seen here 5
With (A, C)

1. The square based on (A, C) is constructed,

2. Then the middle m of $[A, C]$ is constructed,

3. The arc of center m passing through c intersects the half-line $[A, C)$ at x.
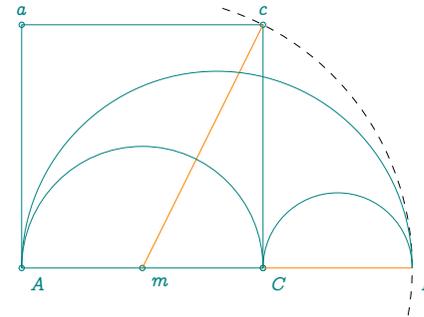
Figure 12: With A and C

With (A, B)

1. The square based on (A, B) is constructed,

2. Then the middle m of $[B, b]$ is constructed,

3. The arc of center m passing through A intersects the half-line $[b, B)$ at x,

4. Finally the line (a, x) cuts the segment $[AB]$ at C.
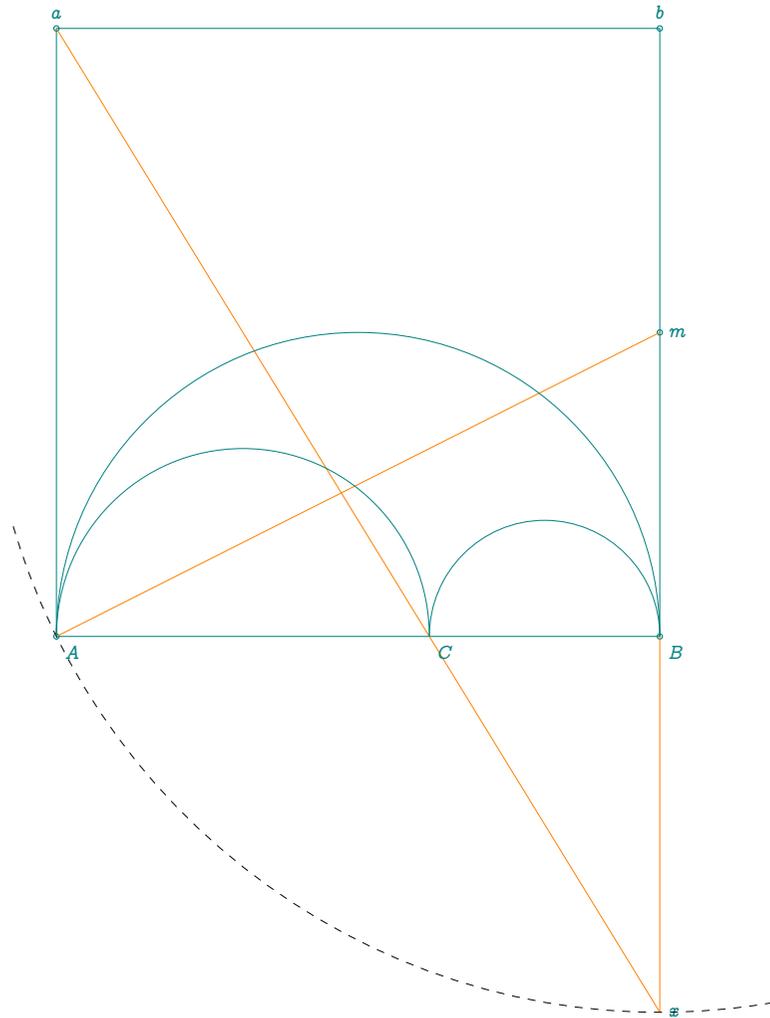
Figure 13: With A and B

2.3.2 Find the golden ratio
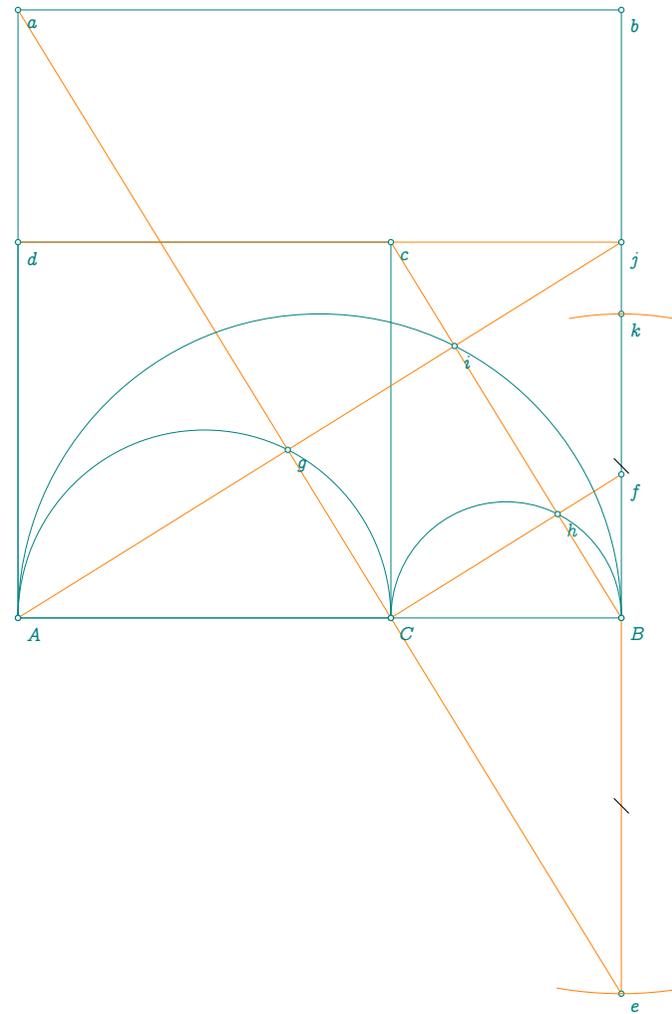
Figure 14: Some devine ratios

$$\frac{AB}{AC} = \frac{Ai}{Ag} = \frac{Bj}{fj} = \frac{Bi}{hi} = \frac{Cf}{Ch} = \varphi$$

2.3.3 Golden section with compass only

Golden section
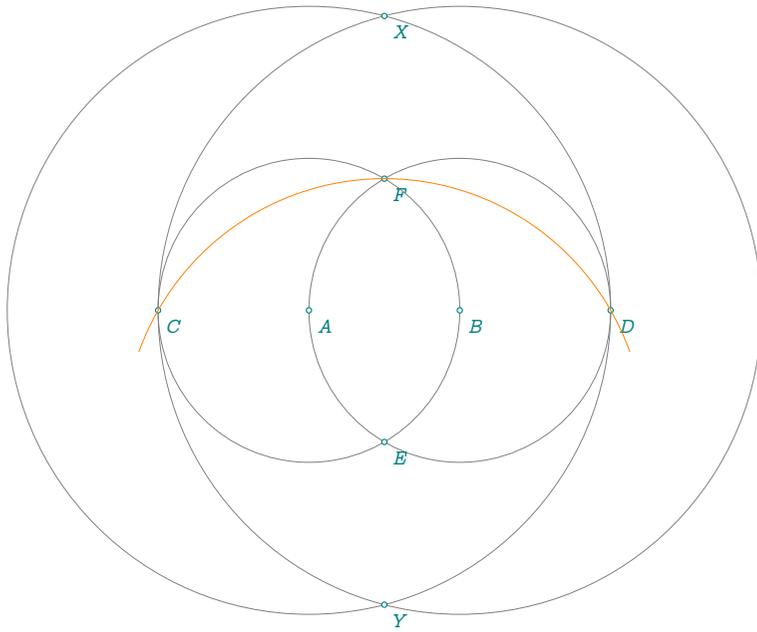
$$\frac{XE}{EF} = 1.6180339887492 \approx 1.6180339887499$$

3 Pappus circle

3.1 First Pappus circle

> **Definition**
>
> *Given an arbelos, there is a circle (δ) included in this arbelos and tangent to (α), (β) and (γ); it is called inscribed circle or circle of Pappus. Archimedes also determined the radius of this circle.*
>
> *The radius of the Pappus circle is $r = \dfrac{ab(a+b)}{a^2 + ab + b^2}$.*

Let us first analyze constructions of this circle in the general case, then we will examine the particular case with the golden arbelos.

Given a golden arbelos $(A, C, B) = (\alpha, \beta, \gamma)$, we now that (β) and γ are external tangent at C and internal tangent at (α) at A, B. Construct the circle (δ) external tangent at $((\alpha), (\beta), (\gamma))$. Let us note P_0, P_1, P_2 the points of tangency.

3.1.1 Construction of Pappus circle with arcs version 1

See paragraph [3.3] for a justification and the document "Simple Constructions of the Incircle of an Arbelos" [Ref: 16].

Without justifications (they will be given later), here is one of the simplest constructions.

1. An arbelos is given the semicircles (α), (β) and (γ) have centers O_0, O_1 and O_2. P and Q are the north poles of (β) and (γ).

2. Two arcs of circles of respective centers P and Q passing through C determine on the arbelos the points of tangency of the circle (δ) with the arbelos: P_0, P_1 and P_2.

3. The center of the Pappus circle is given by the intersection of the lines (O_1P_1) and (O_2P_2)
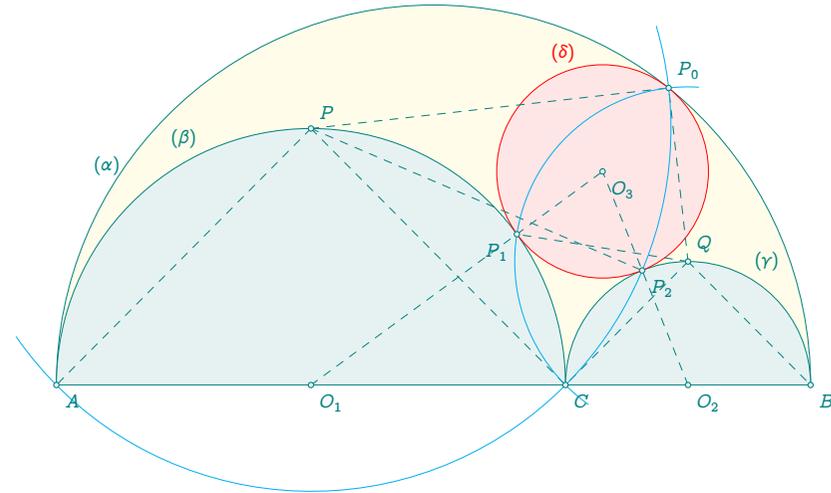
Figure 15: Pappus circle with arcs

3.1.2 Construction of Pappus circle with arcs version 2

See paragraph [3.3] for a justification and the document "Simple Constructions of the Incircle of an Arbelos" [Ref: 16].

Let O be the midpoint of the arc $(\widehat{AB})$. Construct

i) The line (OC) to intersect the semicircle (α) at P_0;

ii) The circle, center O, through A and B, to intersect the semicircles (β) and (γ) at P_1 and P_2.
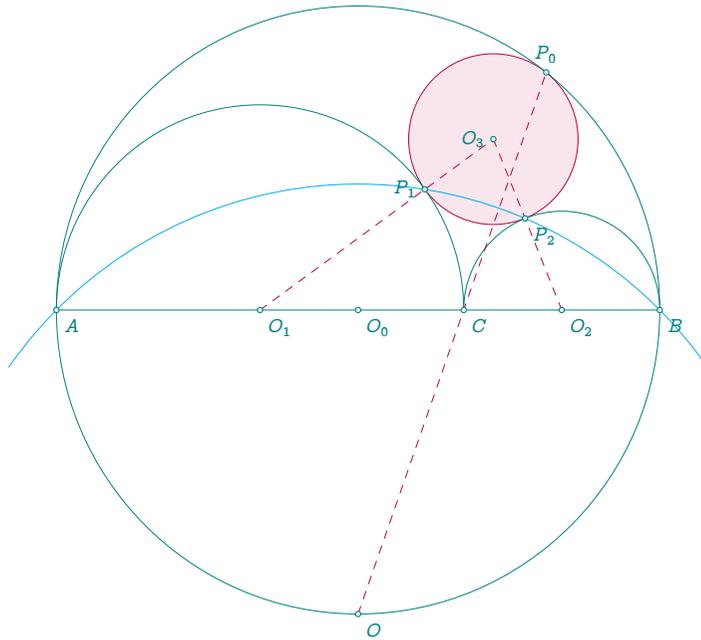
Figure 16: Pappus circle with arcs version 2

Figure 17: Pappus circle with squares

3.2 Determining the radius

Let O_3 be the center of (δ) and H_4 its orthogonal projected onto the line (AB).
H_4 belongs to the segment $[CO_2]$ since we have assumed $a >= b$.
From the right-angled triangles $O_0O_3H_4$, $O_1O_3H_4$ and $O_2O_3H_4$ we get

$$O_3O_0^2 = O_3H_4^2 + O_0H_4^2$$
$$O_3O_1^2 = O_3H_4^2 + O_1H_4^2$$
$$O_3O_2^2 = O_3H_4^2 + O_1H_4^2$$

Either

$$(a+b-r)^2 = O_3H_4^2 + O_0H_4^2$$
$$(a+r)^2 = O_3H_4^2 + (b+O_0H_4)^2$$
$$(b+r)^2 = O_3H_4^2 + (a-O_0H_4)^2$$

because
$$O_1H_4 = O_1O_0 + O_0H_4 = O_0H_4 + (AO_0 - AO_1) = O_0H_4 + (a+b) - a = b + O_0H_4$$
and

3.1.3 Construction of Pappus circle with squares

See the document "Simple Constructions of the Incircle of an Arbelos" [Ref: 16].

$O_2H_4 = O_0O_2 - O_0H_4 = O_1O_2 - O_0O_2 - O_0H = (a+b) - b - O_0H_4 = a - O_0H_4.$

From the first two equalities we get

$b \cdot O_0H_4 = 2ar - b^2 - ab + br$

and from the first and third we get

$a \cdot O_0H_4 = a^2 + ab - ar - 2br$. Hence $r = \dfrac{ab(a+b)}{a^2 + ab + b^2}$.
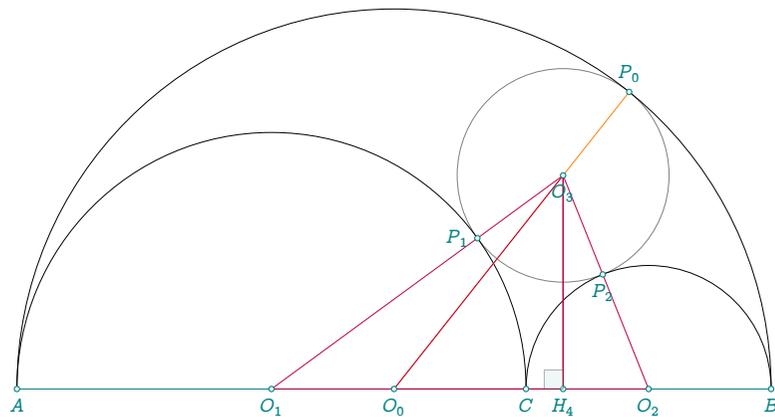
Figure 18: Pappus circle radius

Figure 19: Pappus circle diameter

Question

Let $[AB]$, the diameter of a semicircle, be divided at C so that $AB/AC = k$. Describe semicircles within the first semicircle and on $[AC]$, $[BC]$ as diameters, and suppose a circle drawn touching the all three semicircles. If $[GH]$ be the diameter of this circle, to find relation between GH and AB.

Proof:
Let $[GH]$ be the diameter of the circle which is parallel to (AC), and let the circle touch the semicircles on $[AC]$, $[AB]$, $[BC]$ in P_1, P_0, P_2 respectively.
Then, by Prop. 1 (7.17.1) A, G and P_0 are aligned, as well as P_0, H and B.
For a like reason AP_1 and H are aligned, BP_2 and G are aligned, as also are C P_1 and G, C P_2 and H.
Let (AP_0) meet the semicircle on (β) at I, and let (BP_0) meet the semicircle on (γ) in J. Join (CI), (CJ) meeting (AP_1), (BP_2) in K, L, and let (GL), (HM) produced meet (AB) in F, E respectively.
Now, in the triangle AGB, the perpendiculars from A, C on the opposite sides meet in K. Therefore by the properties of triangles, (GF) is perpendicular to (AC). Similarly (HE) is perpendicular to (BC).
The polygon $EFGH$ is a rectangle. Let's show that it is a square.

1. It is easy to show that the lines (CJ) and (AP_0) are parallel as are the lines (BP_0) and (CI).

    Therefore $\dfrac{AC}{BC} = \dfrac{AL}{LH} = \dfrac{AF}{FE}$, and $\dfrac{BC}{AB} = \dfrac{CM}{MG} = \dfrac{BE}{EF}$.

    Hence $\dfrac{AF}{EF} = \dfrac{EF}{BE}$ (1). (All this is used in proposition 6 of the book of lemmas.)

2. Now we can notice that the right triangles AGF and EBH are similar. Thus $\dfrac{GF}{AF} = \dfrac{BE}{HE} = \dfrac{BE}{GF}$ (2).

3. With (1) and (2) $GF^2 = EF^2$ so $GF = EF$. The rectangle $EFGH$ is a square.

Now suppose that B divides $[AC]$ such as :

$$k = \frac{AC}{CB} \quad \text{then} \quad AF = kEF \quad \text{and} \quad EF = kBE$$

We have

$$AB = AF + FE + EB \quad \text{either} \quad AC + CB = AF + FE + EB$$

or

$$AB = kFE + FE + EF/k = \frac{k^2 + k + 1}{k} EF$$

Finally

$$\frac{AB}{GH} = \frac{k^2 + k + 1}{k}$$

3.3 Points concyclic

> **Lemma: points concyclic**
>
> *Let P and Q be the midpoints of the semicircles (γ) and (β) respectively. Let O be the midpoint of the semicircle (α) on the opposite side of the arbelos. If the incircle of the arbelos is tangent to the semicircles (γ) and (β) and (α) at P_2, P_1, P_0 respectively, then*
>
>   *(i) A, C, P_2, P_0 lie on a circle, center Q;*
>
>   *(ii) B, C, P_1, P_0 lie on a circle, center P;*
>
>   *(iii) The points A, B, P_2 and P_1 lie on a circle, center O.*

Proof: [Peter Y. Woo]

(i) Let us assume that the arbelos and the Pappus circle are constructed. Let D be the intersection of the semicircle (α) with the line perpendicular to (AB) at C. See (Figure 20).

Note that $AB \cdot AC = AD^2$ ($[DC]$ height of the right triangle ADB in D).

Consider the inversion i_A with respect to the circle $\mathcal{C}(A, D)$ of center A that goes through D.

$i_A(B) = C$, and i_A leaves the line AB invariant. The semicircles (β) and (α) have as respective inverses the half-straight lines ℓ' and ℓ perpendicular to (AB) at C and B respectively. The semicircle (γ) is also invariant by i_A because its center belongs to the line (AB). The incircle (δ) or $(P_0 P_1 P_2)$ of the arbelos is inverted into a circle tangent to the semicircle (γ), and the lines ℓ, ℓ' at Q, P_1', P_0' respectively.

Since the semicircle (γ) is invariant, the points A, P_2, and Q are aligned because the image of Q is P_2.

$CBP_1' P_0'$ is a square and Q is its center thus B, Q and P_0' are aligned.

Their inverses are therefore on a circle through A. A, C, P_2 and P_0 are cocyclic.

Let O be the midpoint of the arc $\widehat{AB}$. O, B, Q, P_0' are aligned. The tangent at A to the circle $\mathcal{C}(A, D)$ is parallel to the line (BL) so the diameter coming from A is perpendicular to the line (BL). The center of the circle

$\mathcal{C}(A, D)$ belongs to the line (AL) and to the perpendicular bisector of $[AC]$ so it is the point P.
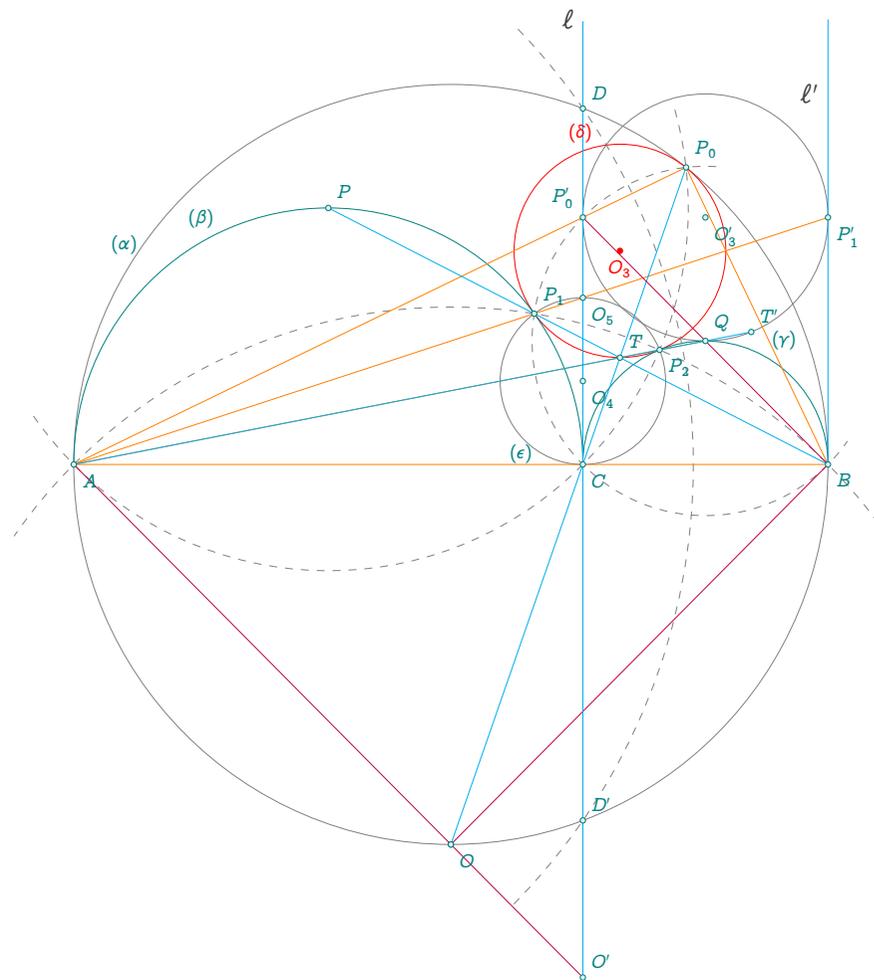
Figure 20: Points Cocyclic

(ii) The same reasoning shows that B, C, P_1, P_0 lie on the circle $\mathcal{C}(Q, B)$ with the inversion i_B center B through D.

(iii) Since C, Q, P_1' are on a line making a 45° angle with (AB), its inversive image by i is a circle through A, B, P_2, P_1 also making a 45° angle with (AB). The center of this circle is necessarily the midpoint O of the semicircle (α) on the opposite side of the arbelos. L belongs to the bisector of $[AB]$ and $\widehat{BAO} = 45°$

Join A, O to intersect the line ℓ at O'. Since $\widehat{BAO'} = 45° = \widehat{BP_0'O'}$, the four points A, P_0', B, O' are cocyclic. Considering the inversive image of the circle, we conclude that the line (CP_0) passes through O.

The center of the incircle can now be constructed as the intersection of the lines joining P_2, P_1, P_0 to the centers of the corresponding semicircles of the arbelos.

See Theorem du Pole Sud[Th 7.22] the line (P_0C) goes through O
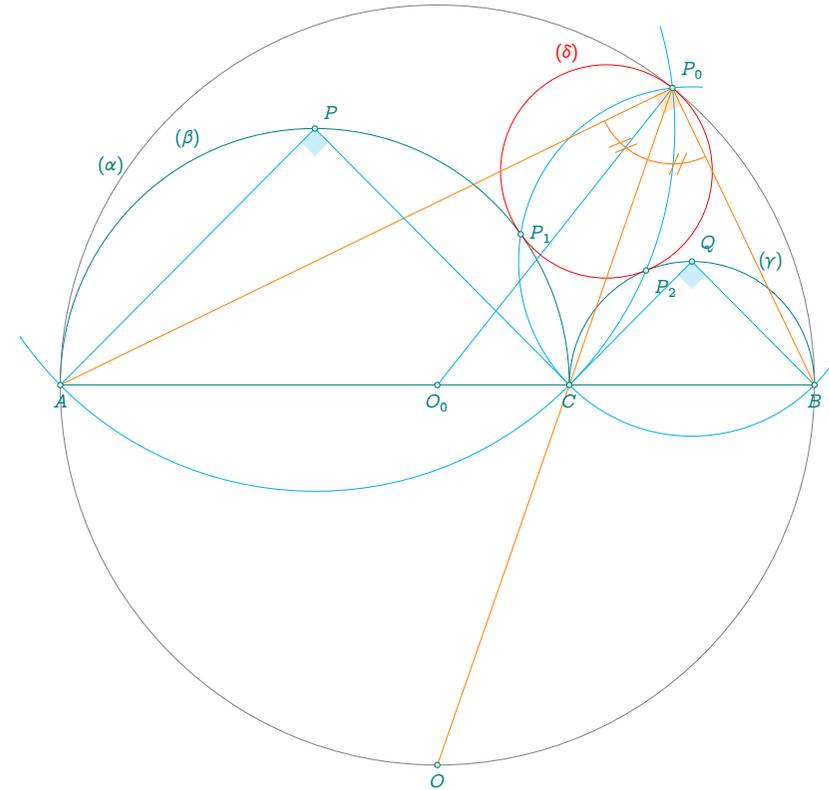
Figure 21: bisector

3.4 Bisector

A, C, P_2 and P_0 are cocyclic (see 3.3) on a circle with center P so $\widehat{AP_0C} = \widehat{APC}/2 = 45°$. In the same way $\widehat{CP_0B} = \widehat{CQB}/2 = 45°$.

+-- Bisector in Arbelos ----------------------------------+
| |
| (P_0C) *is the bisector of* $\widehat{AP_0B}$ |
| |
+--+

(P_0C) is the bisector of $\widehat{AP_0B}$

If O is the midpoint of the arc $\widehat{BC}$ then P_0, C and O are aligned. (Th. South Pole lemma [7.22])

3.5 Midpoints of arcs in an arbelos

> **Midpoints**
>
> *The points A, B and C define an arbelos, (P_0, P_1, P_2) is the pappus circle. The lines $(AQ), (BP)$ and (CO) intersect at a point T midpoint and south pole of the incircle (δ) of the arbelos.*

Proof: Consider the inversion i_A with respect to the circle $\mathcal{C}(A,D)$ of center A that goes through D. The circle (γ) is invariant by i_A so $i_A(P_2) = Q$ and A, P_2, Q are collinear, as are B, P_1, P. In Figure 22, let T be the intersection of the line (AQ) with the circle (δ). The inversive image T' (in the circle $\mathcal{C}(A,D)$) is the intersection of the same line with the circle $QP_1'P_0'$. Note that. $\widehat{AT'P_0'} = \widehat{QT'P_0'} = \widehat{QP_1'P_0'} = 45° = \widehat{ABP_0'}$ so that A, B, T', P_0' are cocyclic.

Considering the inversive image of this circle, we conclude that the line (CP_0) contains T. In other words, the lines (AQ) and (CP_0) intersect at the point T on the circle $(P_0P_1P_2)$. Likewise, (BQ) and (CP_0) intersect at the same point.

Now we need to show that the point T is the south pole of the circle (δ). Consider the triangle GP_0H it has the same bisector P_0C at P_0 as the triangle AP_0B. This bisector passes through the south pole of the circle $P_0P_1P_2$ (δ).

We have another construction of the first Pappus circle. (see figure 23)

- Construct the golden arbelos (A, C, B);

- Construct P and Q the midpoints of the arcs $\widehat{AC}$ and $\widehat{CB}$;

- Construct T the common point of the lines (AQ) and (BP). These lines cuts γ and β at the points P_2 and P_1;

- Construct the common point O_3 of the lines (O_1P_1) and (O_2P_2);

- Draw the circle with center O_3 through P_0. It's the Pappus circle, the first circle of the Pappus chain.
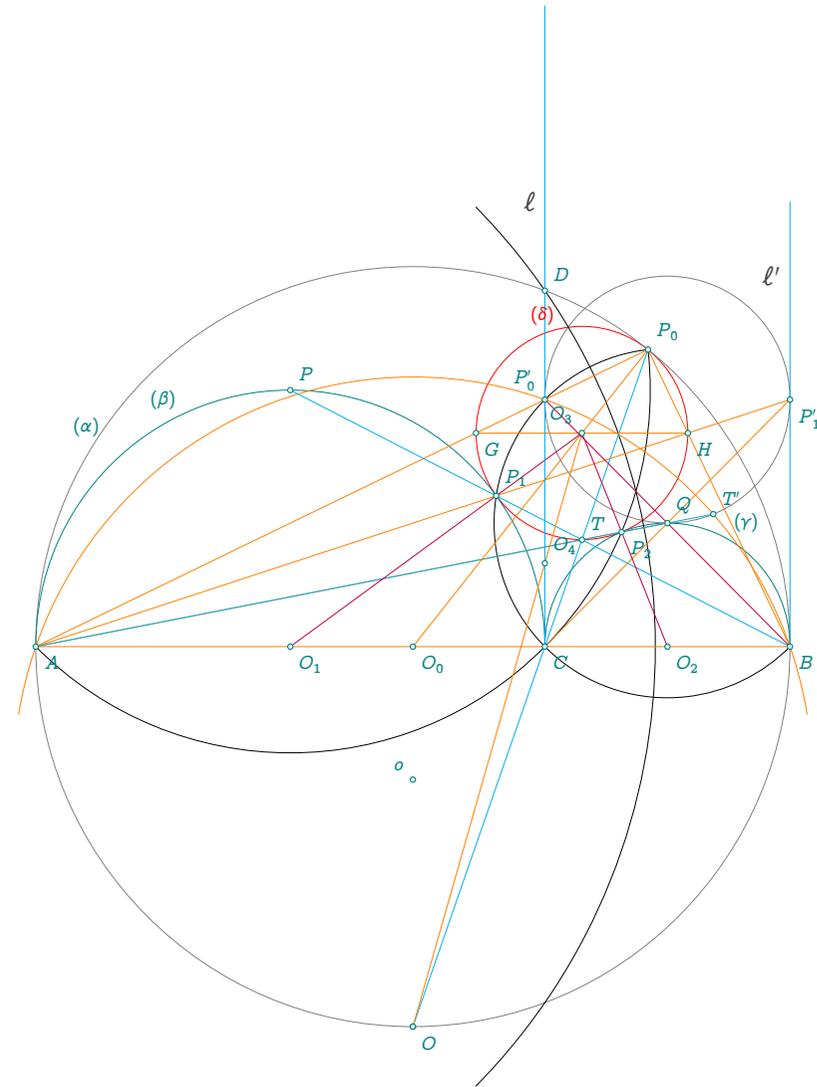
Figure 22: South Pole of Pappus circle T
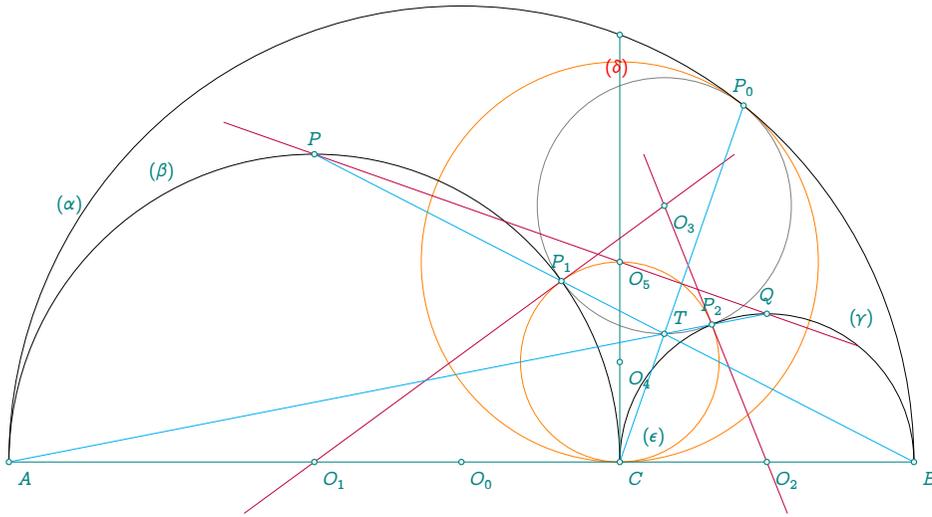
Figure 23: Pappus circle with midpoints

Some properties:

1. The point D is the intersection of the line perpendicular in C to (AB) with the circle (α);

2. The line (PQ) intersects (CD) at O_5 which is the north pole of the Bankoff circle;

3. O_4 is the middle of $[CO_5]$; The circle of diameter $[CO_5]$ is the Bankoff circle which intersects the circles (β) and (γ) at P_1 and P_2;

4. Lines (BP) and (AQ) intersect at T south pole of the circle of Pappus;

5. Line (CT) intersects the circle (α) at P_0.

4 Bankoff Circle and other remarkable circles

4.0.1 Existence of the Bankoff circle

Let's assume an arbelos $((\alpha),(\beta),(\gamma))$ with the first pappus circle (δ). P_0, P_1 and P_2 the points of tangency. O_0, O_1, O_2 and O_3 the centers os the semicircles and circles.

> **Theorem 4.3 (Bankoff)**
>
> *The points of tangency of the incircle of the arbelos with the semicircles (β) and (γ), together with (α), are the points of tangency of the incircle (ϵ) of triangle $O_1O_3O_2$ with the sides of the triangle.*
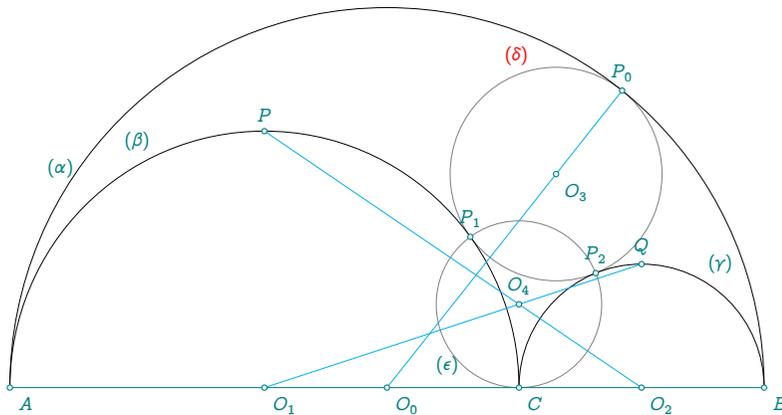
Figure 24: Bankoff Incircle

Proof and onstruction of the figure 24:

Since $O_1P_1 = O_1C$, $O_2P_2 = O_2C$, and $O_3P_2 = O_3P_1$, the points C, P_1, P_2 are the points of tangency of the incircle of triangle $O_1O_2O_3$ with its sides. The semi-perimeter p of the triangle is

$$p = a + b + r = a + b + \frac{ab}{a^2 + ab + b^2} = \frac{(a+b)^3}{a^2 + ab + b^2}$$

Its area, by application of the Heron relation, is

$$\mathcal{A} = \sqrt{\frac{(a+b)^3}{a^2+ab+b^2} \cdot \frac{ab}{a^2+ab+b^2} \cdot a \cdot b} = \frac{ab(a+b)^2}{a^2+ab+b^2}$$

The area of a triangle is also equal to the product of its half-perimeter by the radius of the inscribed circle. Thus we have

$$\mathcal{A} = \rho p = \rho \frac{(a+b)^3}{a^2+ab+b^2} \text{ then } \rho = \frac{ab}{a+b}$$

where ρ is the radius of the circle inscribed on the triangle $O_1O_2O_3$.

Let P and Q be the midpoints of the semicircles (AC) and (CB) respectively.

(i) The lines O_1Q and O_2P to intersect at O_4. The point of intersection O_4 of the lines O_1Q and O_2P verify $CO_4 = \frac{ab}{a+b}$.

(ii) The circle with center O_4, passing through C to intersect the semicircle (AC) at P_1 and (CB) at P_2,

(iii) The lines O_1P_1 and O_2P_2 to intersect at O_3.

The circle with center O_3 passing through P_2 touches the semicircle (β) at P_1 and also the semicircle (α) at P_0.

Another proof:

Let's show that (P, P_1) intersects (ϵ) in B' and (PC) intersects (ϵ) in A'.

$$\widehat{P_1PC} = \frac{1}{2}\widehat{P_1O_1C} = \widehat{P_1O_1O_4} \text{ and } \widehat{P_1B'C} = \frac{1}{2}\widehat{P_1O_4C} = \widehat{P_1O_4O_1}$$

$$\widehat{P_1PC} + \widehat{P_1B'C} = \widehat{P_1O_1O_4} + \widehat{P_1O_4O_1} = 90°$$

$\widehat{A'CB'} = 90°$ so $[A'B']$ is a diameter of ϵ.

$\widehat{PCP'} = 90°$ so P', C, B' are aligned.

A' is the orthocenter of the triangle $PP'B'$ so A' belongs to the line $P'P_1$

Moreover $(A'B')$ is parallel to (AB). $\widehat{CA'O_4} = \widehat{A'CO_1} = 45°$ as base angles of rectangular isosceles triangles.

Remark:

– From Gaspard Monge's "The Three Chords Theorem". the interior tangents common to $(\delta), (\beta), (\gamma)$ in P_2, P_1 and C concur in O_4

(ϵ) is therefore orthogonal to (β) and (γ).

– Let us note P, P' the points of intersection of the perpendicular bisector of $[AC]$ with (β). A', B' the points of intersection $(P'P_1)$, (PP_1) with (ϵ).

From Altshiller-Court "Orthogonal Circles", * $(A'B')$ is a diameter of $epsilon$. $(A'B')//(AB)$. - The circles (ϵ) and (β), the base points P_1 and C, the monian $(B'P_1P)$, the parallels $(A'B')$ and (AB) , lead to Reim's theorem 1' ; consequently, C, A', P are aligned. Also C, B', Q are aligned

– From Altshiller-Court "Orthogonal Circles", applied to orthogonal circles (ϵ) and (β), P', C, B' are aligned. As a result P', C, B', Q are aligned.

According to Thales "Triangle inscribable in a semicircle", $(PC) \perp (QC)$.

– B, C, P_1 and P_0 are cocyclic on a circle of center Q. (PC) is the tangent at C to this circle. Consequently, B', P_1 and B are aligned. P, P_1, B', B are aligned.

4.1 Radius of the Bankoff circle

> **Radius of the Bankoff circle**
>
> *The radius of the Bankoff circle $(\epsilon) = (C, P_2, P_1)$ is equal to the Archimedean circle radius $\dfrac{ab}{a+b}$. We call an archimedean circle any circle constructed from an arbelos whose radius is equal to $\dfrac{ab}{a+b}$. In a golden arbelos this radius is $a - b$.*

Figure 26: Construction of the Bankoff Circle

Proof :

$((\beta), (\gamma), (\alpha), (\delta))$ an arbelos and (δ) the Pappus circle.

We can notice that the Bankoff circle is also the inscribed circle of the triangle $O_1 O_3 O_2$ where O_3 is the center of the Pappus circle (δ), because

$O_1 P_1 = O_1 C = a$

$O_2 P_2 = O_2 C = b$

$r = O_3 P_1 = O_3 P_2 = \dfrac{ab(a+b)}{a^2 + ab + b^2}.$

The half-perimeter of this triangle is equal to $a + b + r = \dfrac{(a+b)^3}{a^2 + ab + b^2}$ and its area, by application of the Heron relation, is

Figure 25: Bankoff Circle 2

$$A = \sqrt{\frac{(a+b)^3}{a^2+ab+b^2} \cdot \frac{ab(a+b)}{a^2+ab+b^2} \cdot a \cdot b} = \frac{ab(a+b)^2}{a^2+ab+b^2}.$$

The area of a triangle is also equal to the product of its half-perimeter by the radius of the inscribed circle. Thus we have

$$A = \rho \frac{(a+b)^3}{a^2+ab+b^2}$$

where ρ is the radius of the circle inscribed in the triangle $O_1O_3O_2$. Hence

$$\rho = \frac{ab}{a+b}.$$

4.2 Sixth circle

Sixth Circle

The common tangent to the circles (β) and (γ) at C intersects the arc $\widehat{BA}$ at D. Let O_5 be the point of the segment $[CD]$ such that $[CO_5]$ is a diameter of the circle (ϵ), i.e. such that $CO_5 = 2\rho = \frac{2ab}{a+b}$. The circle (κ) with center O_5 through C is tangent to (α) at P_0. In a golden arbelos, $\rho = a - b = \frac{b}{\varphi}$.
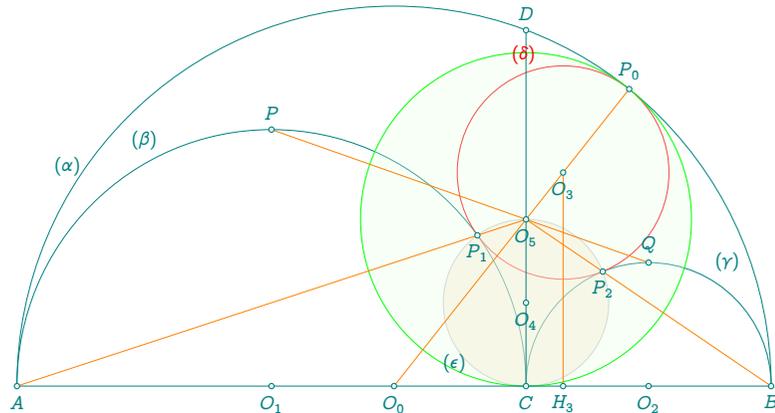

Figure 27: Sixth circle

Proof:

Let H_3 be the orthogonal project of O_3 onto (AB).

$$O_0H_3 = \frac{-2ar}{b} + b + a - r \text{ and } r = \frac{ab(a+b)}{a^2+ab+b^2}.$$

$$O_0H_3 = \frac{2a^2b(a+b)}{b(a^2+ab+b^2)} - b - a + \frac{ab(a+b)}{a^2+ab+b^2}$$

$$O_0H_3 = \frac{2a^3b + 2a^2b^2 - (a+b)^3}{a^2+ab+b^2}$$

$$O_0H_3 = \frac{2a^3b + 2a^2b^2 - a^3 - 3a^2b - 3ab^2 - b^3}{a^2+ab+b^2}$$

$$O_0H_3 = \frac{(a-b)(a+b)^2}{(a^2+ab+b^2)}$$

We deduce $O_0H_3^2 = \frac{(a-b)^2(a+b)^4}{(a^2+ab+b^2)^2}$

In the right-angled triangle $O_0O_3H_3$:

$$O_3H_3^2 = O_0O_3^2 - H_3O_3^2$$

$$O_3H_3^2 = (a+b-r)^2 - \frac{(a-b)^2(a+b)^4}{(a^2+ab+b^2)^2} = \frac{4a^2b^2(a+b)^2}{(a^2+ab+b^2)^2}$$

So $O_3H_3 = \frac{2ab(a+b)}{(a^2+ab+b^2)}$

We finally obtain:

$$\frac{CO_5}{CO_0} = \frac{(ab)}{(a+b)(a-b)} = \frac{2ab(a+b)}{(a-b)(a+b)^2} = \frac{2ab}{(a-b)(a+b)} = \frac{O_3H_3}{O_0H_3}$$

Thus the points O_0, O_5, O_3 and P_0 are aligned.

Moreover in the right triangle O_0CO_5 we obtain $O_0O_5 = \frac{a^2+b^2}{a+b}$ which leads to $O_5P_0 = a + b - \frac{a^2+b^2}{a+b} = \frac{2ab}{a+b} = 2r$. $O_5P_0 = O_5C$ and the circle (κ) is tangent to the circle (α) as well as to the circle (δ) at point P_0.

It's easy to show that O_5 belongs to (PQ).

Let's show that O_5 is on (AP_1) and on (BP_2). O_5 is on (B, M) if you consider the right triangles BMO_2 and BO_5C.
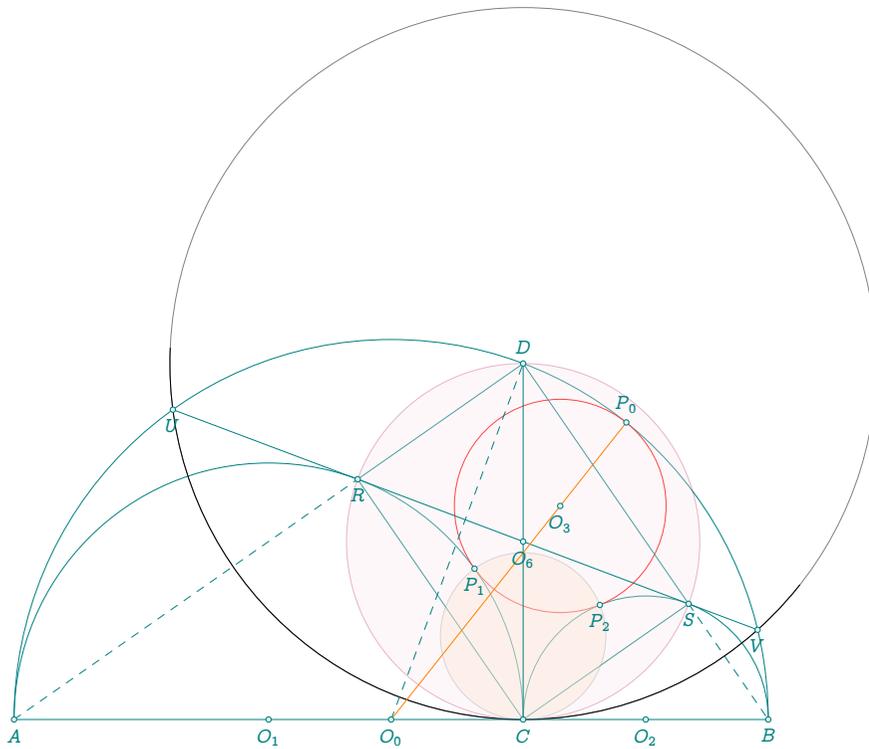
4.2.1 Seventh circle

center D passing through C.

It is easy to show that $(O_0 D)$ is orthogonal to (UV) which proves that D is the middle of the arc $\widehat{UV}$.

Figure 28: Seventh Circle

(AD) and (BD) intersect (β) and (γ) at R and S respectively.

The polygon $CRDS$ is a rectangle (obvious) and the middle of $[CD]$ named O_6 is the center of a circle circumscribing the rectangle.

Let us show that (RS) is a common tangent to the circles (β) and (γ).

Consider the inversion of I_D and power DC. The circles (β) and (γ) are globally invariant because they are orthogonal at C. $I_D(A) = R$ and $I_D(B) = S$. The line (RS) is the image by I_D of the circle (α) tangent at A and B to the circles (β) and (γ). Thus (RS) is tangent to the circles (β) and (γ) at R and S.

The line (RS) intersects the circle (α) at U and V. U and V belong to the circle (α) and to its image, so they are invariant; hence they belong to the circle with

5 Midcircles

> **Definition**
>
> *From Eric Danneels and Floor van Lamoen: A midcircle of two given circles is a circle that swaps the two given circles by inversion. Midcircles are in the same pencil of circles as the given circles. The center of the midcircle(s) is one or both of the centers of similitude. We can distinguish four cases:*
>
> *(i) The two given circles intersect: there are two midcircles with centers at the centers of similitude of the given circles;*
>
> *(ii) One given circle is in the interior of the other given circle. Then there is one midcircle with center of similitude at the internal center of similitude of the given circles;*
>
> *(iii) One given circle is in the exterior of the other given circle. Then there is one midcircle with center at the external center of similitude of the given circles. Clearly the tangency cases can be seen as limit cases of the above;*
>
> *(iv) If the circles intersect in a single point, the unique midcircle has center at the external similitude center or at internal similitude center.*

Let's look at each of these cases:

(i) If the two given circles intersect, then there are two circles of inversion through their common points, with centers at the centers of similitudes. The two midcircles bisect their angles and are orthogonal to each other. The centers of the midcircles are the internal center of similitude and the external center of similitude I and J.

Consider two intersecting circles $(\mathcal{A})$ and $(\mathcal{B})$. We can obtain the centers of similarity of these two circles by constructing EH and FG two diameters parallel of the circles $(\mathcal{A})$ and $(\mathcal{B})$. The line (GE) intercepts the line (AB) in J and the line (EF) intercepts the line (AB) in I. The circles $(\mathcal{I})$ and $(\mathcal{J})$ are orthogonal and are the midcircles of $(\mathcal{A})$ and $(\mathcal{B})$. The division $(A, B; I, J)$ is harmonic.
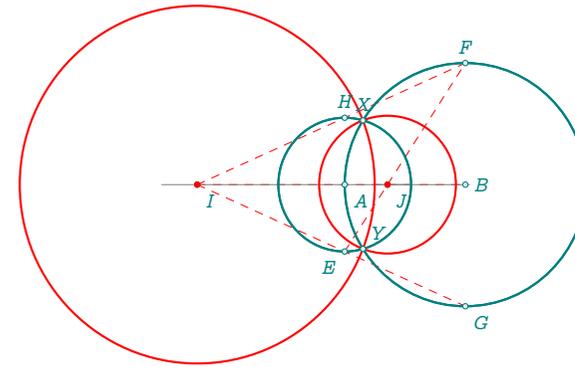

Figure 29: Given circles intersect

(ii) One given circle is in the interior of the other given circle.
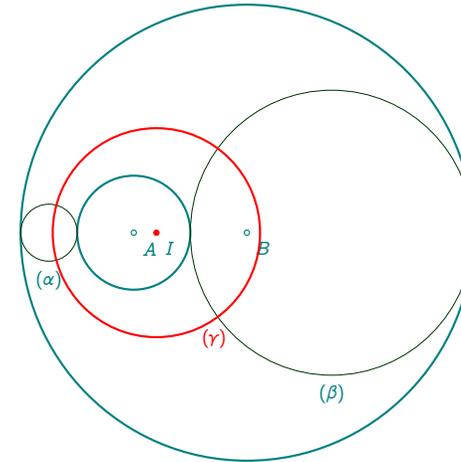

Figure 30: Given circle is in the interior

This case is a little more complicated. We'll construct the two circles (α) and (β) tangent to the two given circles. Then we construct the radical circle orthogonal to the circles (α) and (β). Its center is the radical center as well as the center of internal similtude of circles of center A and B.

(iii) When the two given circles are external to each other, we construct the external center of similitude of the two given circles. I is the center of external similarity of the two given circles. To obtain the inversion circle, simply note that H is such that $IH^2 = IE \times IF$
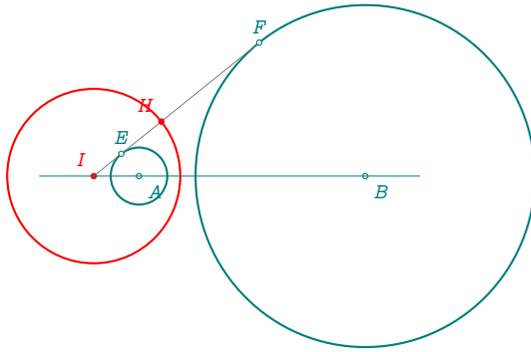

Figure 31: Given circles are external to each other

Figure 33: Fig. b

(iv) Consider two tangent circles $(\mathcal{A})$ and $(\mathcal{B})$,

– $(\mathcal{B})$ being external and angent to $(\mathcal{A})$. The construction is identical to the previous one.
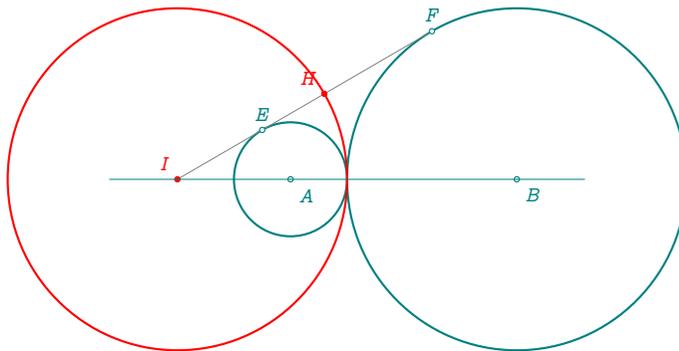

Figure 32: Fig. a

– When one of the given circles is inside and tangent to the other, the construction is easy (see figure 33).
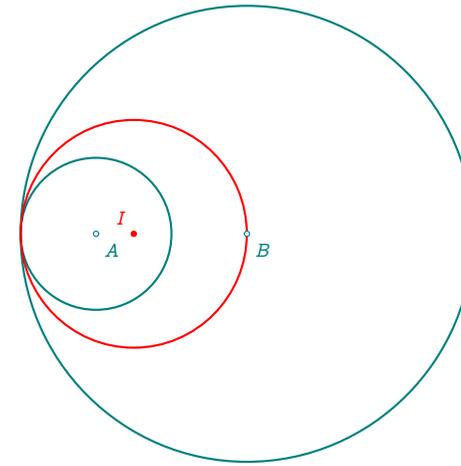
5.1 Application to the arbelos

5.1.1 The pencil of midcircles

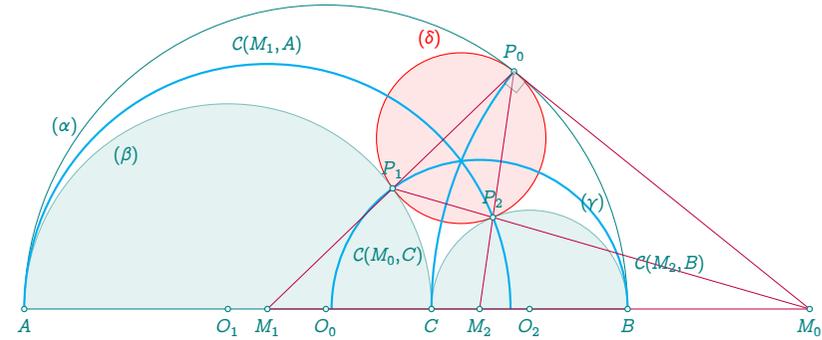Figure 34: Midcircles

Let be M_0, M_1 and M_2 the centers of the midcircles of (β) and (γ), (α) and (β) and finally (α) and (γ). These circles have M_0, M_1 and M_2 as centers and pass through C, A and B. They can be defined as external or internal centers of similitude.

(i) $\mathcal{C}(M_0, C)$ this circle passes this C and P_0.

(ii) $\mathcal{C}(M_1, A)$ this circle passes through A and P_2.

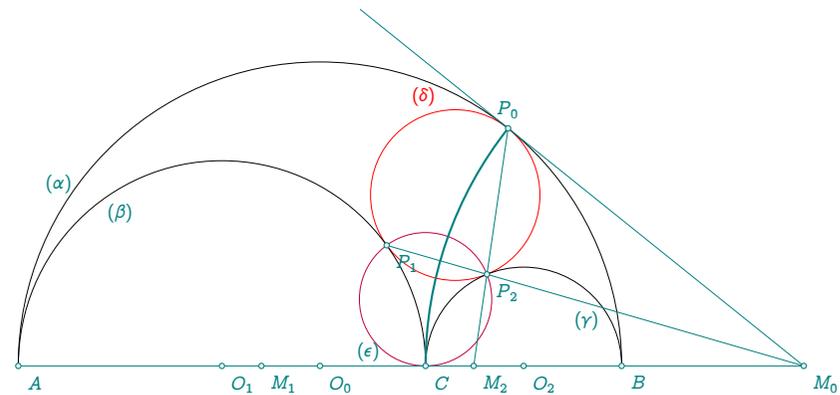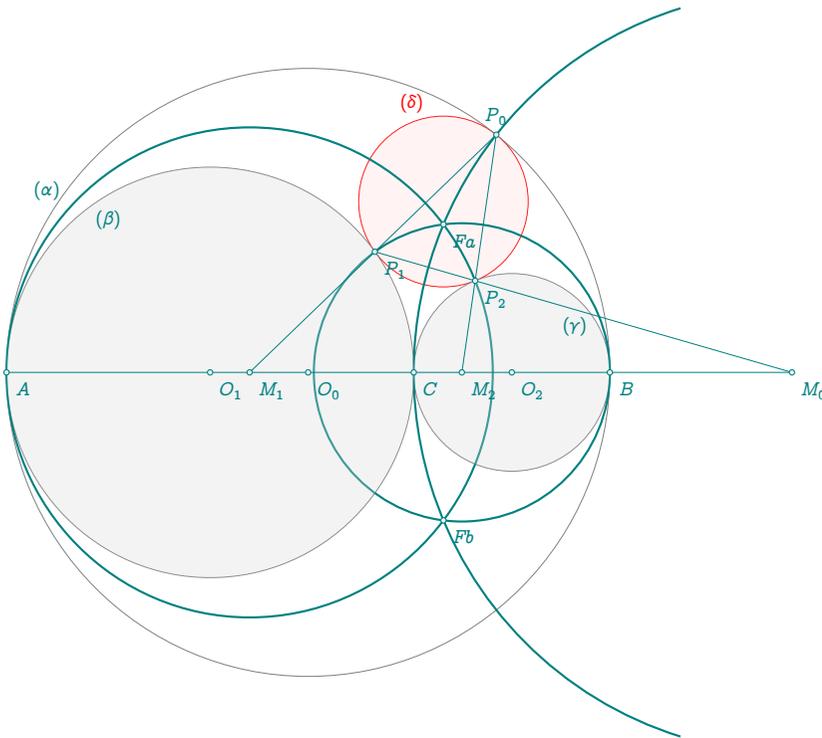(iii) $\mathcal{C}(M_2, B)$ this circle passes this A and P_1.

Figure 35: Midcircles fig 1

(i) Let's call $\mathcal{I}_0$ the inversion of pole M_0 with respect to the circle $C(M_0, C)$.

The circle (γ) has for image the circle (β) with $\mathcal{I}_0(B) = A$ and $\mathcal{I}_0(C) = C$. The Bankoff circle $(\epsilon) = C(O_4, C)$ is globally invariant because it is tangent at C to the globally invariant line (AB). The image of P_2 is a (ϵ) point and a (β) point, which is P_1. The image of P_2 is a (ϵ) point and a (β) point, which is P_1.

P_2 and P_1 are points of (δ), so (δ) is also a globally invariant circle. P_0 is a point belonging to two globally invariant circles (α) and (δ), it is therefore invariant and belongs to the inversion circle: $P_0 \in C(M_0, C)$.

The first consequence is that $M_0 P_0 = M_0 C$. Furthermore, the line $M_0 P_0$ is tangent to (δ) and (α) (a single point in common).

For M_0 the external center of the similitude defined by the circles (β) and (γ) can be obtained by the harmonic division $(O_1, O_2; C, M_0) = -1$.

We deduce that $M_0 P_0^2 = M_0 C^2 = M_0 B \times M_0 A$. By noticing that $M_0 C = M_0 B + BC = M_0 B + 2b$ and $M_0 A = M_0 B + BA = M_0 B + 2a + 2b$, it is easy to establish that

$$BM_0 = \frac{2b^2}{a-b} \text{ and } AM_0 = \frac{2a^2}{a-b}$$

Let us show that this implies that $(O_1, O_2; C, M_0)$ is a harmonic division.

The division $(O_1, O_2; C, M_0)$ will be harmonic if and only if

$\dfrac{O_1 M_0}{O_2 M_0} = \dfrac{O_1 C}{O_2 C}$ or $\dfrac{O_0 M_0}{O_1 M_0} = \dfrac{a}{b}$

$O_0 M_0 = 2b + a + BM_0$ and $O_1 M_0 = b + BM_0$ we obtain

$\dfrac{2b + a + BM_0}{b + BM_0} = \dfrac{a}{b}$ and therefore $BM_0 = \dfrac{2b^2}{a-b}$.

(ii) Let be M_1 the internal center of similitude of (α) and (β). Let's call $\mathcal{I}_1$ the inversion of pole M_1 with respect to the circle $C(M_1, A)$.

We have $\mathcal{I}_1(A) = A$ and $\mathcal{I}_1(C) = B$. Let's show that $\mathcal{I}_1(P_1) = P_0$.

The circles (β) and (δ) are globally invariant by $\mathcal{I}_1$ (tangent to the circles (α) and (β)).

The inversion by $\mathcal{I}_1$ of P_1 is a point of (α) and (δ), so we deduce that $\mathcal{I}_1(P_1) = P_0$. The points M_1, P_1 and P_0 are aligned.

Moreover, the image of P_2 is a point on both globally invariant circles, so it is invariant and $\mathcal{I}_1(P_2) = P_2$. Thus P_2 is a point of $C(M_1, A)$.

Consider the case of M_1 taking A as origin, then, the definition of the center of internal similarity:

$$AM_1 = \frac{AO_0 \times AO_1 + AO_1 \times AO_0}{AO_0 + AO_1} = \frac{2a(a+b)}{2a+b}$$
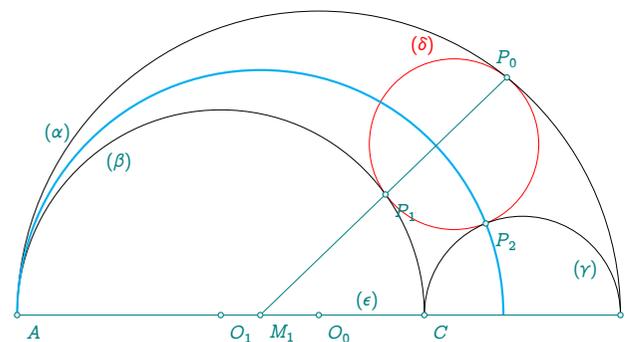

Figure 36: Midcircles fig 2

We can obtain the points M_1 using the following harmonic division $(O_0, O_1; M_1, A) = -1$.

The harmonic division $(O_0, O_1; M_1, A) = -1$ results in

$\dfrac{M_1 O_0}{M_1 O_1} = \dfrac{AO_0}{AO_1} = \dfrac{a+b}{a}$ or $\dfrac{M_1 O_0}{M_1 O_1} = \dfrac{AO_0 - AM_1}{AM_1 - AO_1} = \dfrac{a+b - AM_1}{AM_1 - a}$ which implies

$$AM_1 = \frac{2a(a+b)}{2a+b}.$$

(iii) Let's call $\mathcal{I}_2$ the inversion of pole M_2 with respect to the circle $C(M_2, B)$. Consider the case of M_2 taking A as origin, then, the definition of the center of internal similarity:
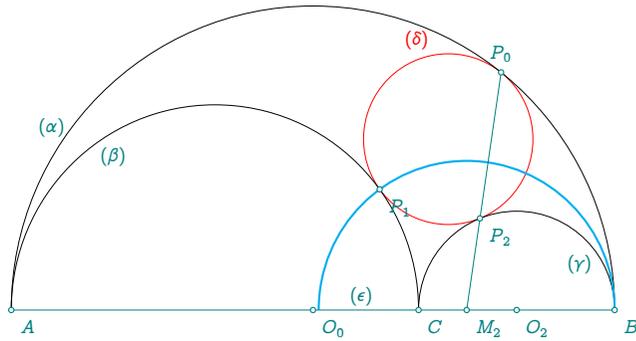
Figure 37: Midcircles fig 3

5.1.2 Ayme pappus

Figure 38: Pappus by Ayme

Consider the case of M_2 taking A as origin, then

$$AM_2 = \frac{bAO_0 + (a+b)AO_2}{a+2b} = \frac{b(a+b) + (a+b)(2a+b)}{a+2b}$$

$$AM_2 = \frac{2a^2 + 4ab + 2b^2}{a+2b} = \frac{2(a+b)^2}{a+2b}$$

$$AM_2 = \frac{2(a+b)^2}{a+2b}$$

We can obtain the points M_2 using the following harmonic division $(O_0, O_2; M_2, B) = -1$.

In the same way as above, we can show that the points M_2, P_2 and P_0 are aligned. Also that P_1 is invariant and a point on the circle $\mathcal{C}(M_2, B)$.
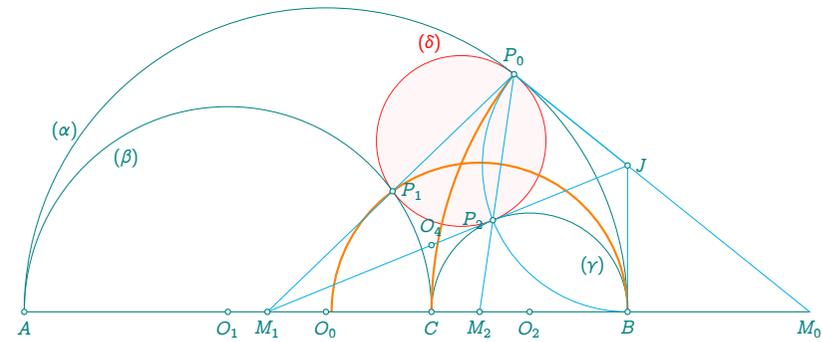
Remark: Let T_B be the tangent at B to the circles (α) and (γ), J the point of intersection of T_B and $(M_1 P_2)$. P_0 is the point of contact of (α) with (δ). T the tangent to (α) and (δ) at P_0.
According to Gaspard Monge "The three Chords theorem" T passes through J
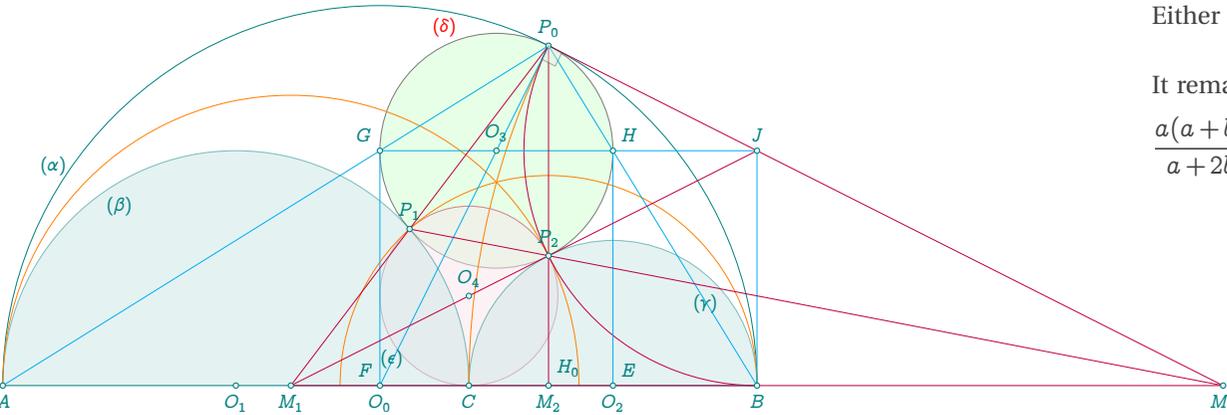
6 Golden

6.1 $H_0 = M_2$

Figure 39: Midcircles

H_0 ? defined by $(A, B; H_0, M_0) = -1$ is such that $P_0 H_0$ is orthogonal to (A, B). Indeed $(P_0 B)$ is the internal bisector of $H_0 P_0 M_0$ and $(A P_0)$ is the external bisector. The line (PM) is tangent to the circle at P. If H_0 is defined by $(A, B; H_0, M_0) = -1$ then the line $P_0 H_0$ is orthogonal to the line (AB).

If, moreover, M_0 defined by $(O_1, O_2; C, M_0) = -1$ is the external center of similarity of the circles of center O_1 and O_2 passing through C. We proved that the tangent at P_0 passes through M_0 and then the line $P_0 H_0$ is orthogonal to the line (AB).

$(O_0, O_2; M_2, B) = -1$ defines the center of internal similarity of the circles of center O_0 and O_2 passing through B. We know that M_2, P_2 and P_0 are aligned.

Let us show that in the case of the **golden arbelos** H_0 and M_2 are identical.

The harmonic division $(A, B; H_0, M_0) = -1$ can be translated by:
$O_0 B^2 = O_0 H_0 \times O_0 M_0$ sachant que O_0 est le milieu de $[AB]$.

It follows that $O_0 H_0 = \dfrac{(a+b)^2}{3a+b}$

The harmonic division $(O_0, O_2; M_2, B) = -1$ can be translated by :

$$\frac{2}{O_0 O_2} = \frac{1}{O_0 M_2} + \frac{1}{O_0 B}$$

Either $\dfrac{1}{O_0 M_2} = \dfrac{2}{a} - \dfrac{1}{a+b}$ and therefore $O_0 M_2 = \dfrac{a(a+b)}{a+2b}$.

It remains to prove that in the case of **golden arbelos**, we have $\dfrac{(a+b)^2}{3a+b} = \dfrac{a(a+b)}{a+2b}$.

A first simplification gives $\dfrac{a+b}{3a+b} = \dfrac{a}{a+2b}$

This can be written: $(a+b)(a+2b) = a(3a+b)$ soit $a^2 + 3ab + 2b^2 = 3a^2 + ab$, which gives

$a^2 = ab + b^2$ or even $\dfrac{a}{b} = \dfrac{a+b}{a}$ which is true when $\dfrac{a}{b} = \varphi$.

Remark: the rectangle $O_0 GJB$ is a gold rectangle.

6.2 Particularities of the golden arbelos

> **Question**
>
> *Let $[AB]$, the diameter of a semicircle, be divided at C so that $AB/AC = k = \varphi$. Suppose a circle drawn touching the all three semicircles. If $[GH]$ be the diameter of this circle, then show that $AB = 2\varphi GH$, $GH = a$, $E = O_2$ and $F = O_0$.*
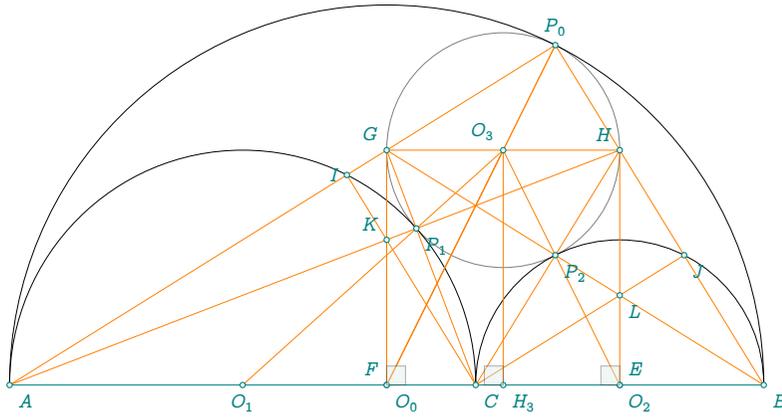
Figure 40: Pappus circle diameter

$$AB = AF + FE + BE = \varphi^2 BE + \varphi BE + BE = (\varphi^2 + \varphi + 1)BE = 2(\varphi + 1)BE$$

$$2(a + b) = 2(\varphi + 1)BE$$

$$b(\varphi + 1) = (\varphi + 1)BE$$

so $BE = b$ and $E = O_2$
Let us show that $F = O_0$

$$\frac{AO_2}{AO_0} = \frac{2a + b}{a + b} = 1 + \frac{a}{a + b} = 1 + \frac{1}{\varphi} = \varphi$$

$AO_2 = \varphi AO_0$ either $AE = AF + FE = (1 + \frac{1}{\varphi}AF) = \varphi AO_0$

Finally $\varphi AF = \varphi AO_0$ and therefore $F = O_0$
$GH = EF = O_0 O_2 = a$.

Proof:
Let $\varphi = \dfrac{a}{b}$ the ratio between AB and GH was given here 3.2

$$\frac{AB}{GH} = \frac{k^2 + k + 1}{k}$$

or, if $\varphi = \dfrac{a}{b}$

$$\frac{AB}{GH} = \frac{\varphi^2 + \varphi + 1}{\varphi} = \varphi + 1 + \frac{1}{\varphi} = 2\varphi$$

The problem here is to find the ratios often equal to the golden ratio.
$AP_0 / AI = \varphi$
Therefore

$$\frac{AC}{CB} = \frac{AK}{KH} = \frac{AF}{EF} \quad \text{and} \quad \frac{BC}{CA} = \frac{BM}{MG} = \frac{BE}{EF}$$

hence

$$\frac{AF}{EF} = \frac{EF}{BE} \quad \text{so} \quad EF^2 = AF \times BE$$

Some consequences:

1. $O_0 O_2 O_4$ is an isosceles triangle with $O_0 O_4 = O_2 O_4 = \dfrac{a\sqrt{5}}{2}$. This implies

   that $O_0 P_0 = \dfrac{a\sqrt{5}}{2} + \dfrac{a}{2} = a\varphi$.

2. $O_2 P_2 = O_2 O_4 - O_4 P_2 = \dfrac{a\sqrt{5}}{2} - \dfrac{a}{2} = \dfrac{a}{\varphi}$

3. $\dfrac{O_0 B}{O_0 G} = \dfrac{a + b}{a} = \varphi$. $FGJB$ is a gold rectangle.

4. In the triangle $M_1 J B$ we have $\dfrac{BJ}{P_2 M_2} = \dfrac{M_1 B}{M_1 M_2}$. Either $BJ = \dfrac{M_1 B \times P_2 M_2}{M_1 M_2}$

   but in the triangle $O_0 GB$ we have $\dfrac{O_0 G}{P_2 M_2} = \dfrac{O_0 B}{M_2 B}$ which can be written

   $\dfrac{a}{P_2 M_2} = \dfrac{a + b}{M_2 B}$ or even $P_2 M_2 = \dfrac{a \times M_2 B}{a + b}$.

   Finally $BJ = \dfrac{M_1 B \times \dfrac{a \times M_2 B}{a + b}}{M_1 M_2}$.

   We know that $M_1 B = 2a$, $M_2 B = \dfrac{2b(a + b)}{a + 2b}$ and

   $M_1 M_2 = M_1 B - M_2 B = 2a - \dfrac{2b(a + b)}{a + 2b}$ then

$$\frac{BM_2}{M_1M_2} = \frac{\varphi}{2}.$$

We can conclude that $BJ = \dfrac{2a^2 \times M_2B}{(a+b)M_1M_2} = \dfrac{a^2 \times \varphi}{a+b} = a$.

We knew that $BM_0 = 2a$ here 6.8 so it is easy to obtain the point M_0 with the compass. Then P_0 is on the circle (α) and on the circle of center BM_0 passing through C.

6.3 Pappus Radius

$EFGH$ is a square and we know $O_0 = F \quad O_2 = E$, so $EF = BG = O_0O_2 = a$
We can find the radius of the Pappus Circle with:

$$r = \frac{ab(a+b)}{a^2+ab+b^2} = \frac{a(\varphi+1)}{\varphi^2+\varphi+1} = \frac{a}{2}$$

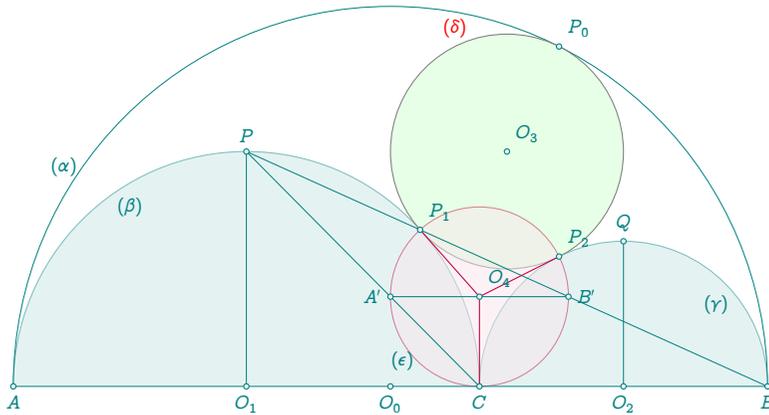### 6.4 Bankoff circle with golden arbelos
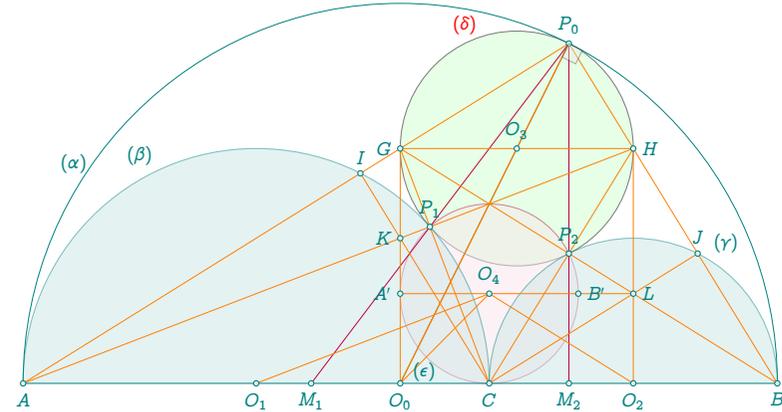

Figure 41: Golden Bankoff Circle

6.4.1 Bankoff Radius

$$\frac{O_5C}{PO_1} = \frac{O_2C}{O_2O_1} = \frac{O_5C}{a} = \frac{b}{a+b}$$

Let $\rho = O_5C$ we deduce from the previous equation that $\rho = \dfrac{ab}{a+b}$

$$\rho = \frac{ab}{a+b} = \frac{b}{1+1/\varphi} = \frac{b}{\varphi} = \frac{a}{\varphi^2} = \frac{b^2}{a}$$

6.5 Alignments

1. O_0, A', K and G are aligned.

   We know that O_0, K and G are aligned, then $A'O_4 = \dfrac{b}{\varphi}$.

   $O_0C = a - b = b\varphi - b = b(\varphi - 1) = \dfrac{b}{\varphi} = A'O_4$ so O_0, A', K and G are aligned.

2. A', O_4, B' and L are aligned.

   L is the orthocenter of the triangle CHB.

   $$\frac{O_2M}{O_0G} = \frac{O_2B}{O_0B} = \frac{b}{a+b}$$

   $$\frac{O_2M}{GH} = \frac{b}{a+b}$$

   $$O_2L = \frac{b}{a+b} \times \frac{2ab(a+b)}{a^2+ab+b^2} = \frac{2ab^2}{a^2+ab+b^2} = \frac{a}{\varphi^2+\varphi+1}$$

   $$O_2L = \frac{2a}{\varphi^2+\varphi+1} = \frac{a}{\varphi^2} = \frac{b}{\varphi} = \rho$$

   A', O_4, B' and L are aligned.

From 4.2 we know that O_5 the north pole of the circle (ϵ) is the center of the circle (κ)

6.6 Special segments

In the case of the **golden arbelos** $F = O_0$ with F projection of G on (AB).

The segment $[BG]$ has length $BG = \sqrt{O_0G^2 + BO_0^2} = \sqrt{(a+b)^2 + a^2} = a\sqrt{\varphi^2 + 1} = a\sqrt{\varphi + 2}$.

Remark: $BP_2 = \sqrt{2}M_2P_2$. $O_0O_4 = O_2O_4 = \dfrac{\sqrt{5}a}{2}$ and $O_0P_0 = \dfrac{\sqrt{5}a}{2} + \dfrac{a}{2} = \dfrac{a\varphi}{2}$

We have seen that $AM_2 = \dfrac{2(a+b)^2}{a+2b}$ which implies $BM_2 = \dfrac{2b(a+b)}{a+2b}$.

$BM_2 = \dfrac{2b(\varphi+1)}{\varphi+2}$

$\dfrac{BM_2}{BO_2} = \dfrac{M_2P_2}{O_2M}$ which implies $BM_2 = \varphi M_2 P_2$

so $M_2P_2 = \dfrac{2b(\varphi+1)}{\varphi(\varphi+2)}$

$\dfrac{BM_2}{M_1M_2} = \dfrac{\varphi}{2}$.

This is because $AM_1 = 2b$, $AM_2 = \dfrac{2(a+b)^2}{a+2b}$ and $BM_2 = \dfrac{2b(a+b)}{a+2b}$.

We deduce $M_1M_2 = \dfrac{2(a+b)^2}{a+2b} - 2b = \dfrac{2(a^2+ab-b^2)}{a+2b}$ then

$\dfrac{BM_2}{M_1M_2} = \dfrac{2b(a+b)(a+2b)}{2(a+2b)(a^2+ab-b^2)} = \dfrac{b(a+b)}{(a^2+ab-b^2)}$

$\dfrac{BM_2}{M_1M_2} = \dfrac{\varphi+1}{\varphi^2+\varphi-1} = \dfrac{\varphi^2}{2\varphi} = \dfrac{\varphi}{2}$

Relations with centers:

$O_0C = a-b = b(\varphi-1)$, $O_2B = b$, $O_0B = a+b = b\varphi^2$, $O_0O_1 = b$ and $O_0O_2 = a = b\varphi$

Relations with centers of the midcircles (M_1, M_2, M_0):

$AM_1 = 2b$, $AM_2 = \dfrac{2(a+b)^2}{a+2b}$, $M_1M_2 = \dfrac{2(a^2+ab-b^2)}{a+2b}$

Relationship with the tangency points

6.7 Golden triangle in the golden Arbelos

Identifying golden and equilateral triangles that arise from the golden Arbelos
Article in The Mathematical Gazette · October 2019

6.7.1 Golden triangle

In order to simplify the demonstrations, we will use the following notations:
$AC = 2a$ and $CB = 2b$
O_0, O_1 and P_2 being the centers of the semi-circles which constitute the arbelos:
$AO_0 = a+b$, $O_0C = a-b$, $O_1O_0 = b$, $O_1O_2 = 2a+2b-a-b = a+b$ and finally $O_2O_0 = b+(a-b) = a$.
C dividing AB in a divine proportion we can write :
$\dfrac{AC}{BC} = \dfrac{a}{b} = \dfrac{a+b}{a} = \varphi$

We want to prove that the triangle O_0MC is golden.
To demonstrate the property we prove that $\dfrac{O_0M}{O_0C} = \varphi$

$\dfrac{AB}{AC} = \varphi$ then $AC = \dfrac{AB}{\varphi} = \dfrac{2(a+b)}{\varphi}$

$O_0C = a-b = AC - AO_0 = \dfrac{2(a+b)}{\varphi} - (a+b) = \dfrac{2(a+b)}{\varphi} - \dfrac{\varphi(a+b)}{\varphi}$

$O_0C = \dfrac{2\varphi(a+b) - (a+b)}{\varphi} = \dfrac{(2\varphi-1)(a+b)}{\varphi} = \dfrac{(a+b)}{\varphi^3}$ (see 2.2.2)

$$O_0C = \dfrac{a+b}{\varphi^3} = \dfrac{(a+b)(2-\varphi)}{\varphi}$$

$O_1L = a = \dfrac{a+b}{\varphi}$

$O_0I = \dfrac{a-b}{2}$

$O_1I = O_1O_0 + O_0I = b + \dfrac{a-b}{2} = \dfrac{a+b}{2}$

$O_0M^2 = O_0I^2 + IM^2 = O_1M^2 - O_1I^2 + O_0I^2$

$O_0M^2 = \dfrac{(a+b)^2}{\varphi^2} - \dfrac{(a+b)^2}{4} + \dfrac{(a+b)^2(2-\varphi)^2}{4\varphi^2}$

$O_0M^2 = \dfrac{4(a+b)^2 - \varphi^2(a+b)^2 + (a+b)^2(2-\varphi)^2}{4\varphi^2}$

$$O_0M^2 = \frac{(a+b)^2[4-\varphi^2+(2-\varphi)^2]}{4\varphi^2}$$

$$O_0M^2 = \frac{(a+b)^2(2-\varphi)}{\varphi^2} = \frac{(a+b)^2}{\varphi^4} \quad O_0L = \frac{a+b}{\varphi^2}$$

With the result 8, $\dfrac{O_0M}{O_0C} = \varphi$, so the triangle O_0MC is golden.

Now let's prove that O_1MC is golden.

$$O_1L = \frac{a+b}{\varphi} \text{ and } O_0C = \frac{a+b}{\varphi^2}$$

$\dfrac{O_1M}{MC} = \varphi$ so the triangle O_1MC is golden too.

The line (O_0M) intersects the circle β at N. Let us draw the line parallel to (MC) passing through N. It cuts $[AB]$ at P. Let us show that $P = O_2$ and that the triangle O_0NO_2 is a golden triangle.

We have $\dfrac{O_0N}{O_0M} = \dfrac{O_0P}{O_0C}$. Either $(a+b) \times \dfrac{\varphi^2}{a+b} = \dfrac{O_0P \times \varphi^3}{a+b}$

Then $O_0P = \dfrac{a+b}{\varphi} = a$ also $P = O_2$.

The triangle O_0NO_2 is isosceles and the angle $\widehat{O_2O_0N} = 72°$ so it is a golden triangle or $\dfrac{O_0N}{O_0O_2} = \dfrac{a+b}{a} = \varphi$ and the conclusion is identical.

6.7.2 Regular pentagons in golden Arbelos
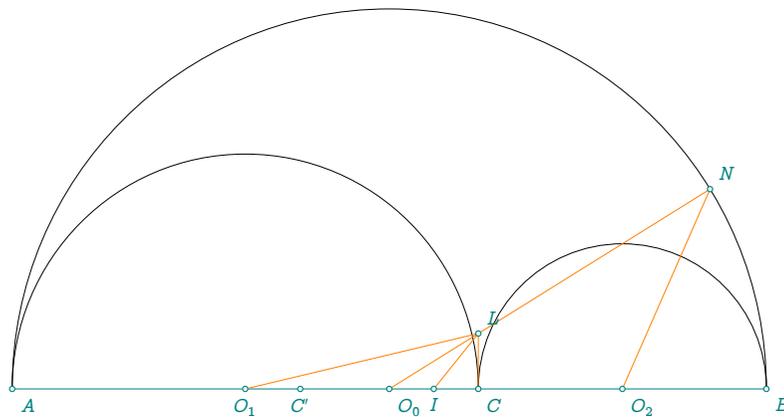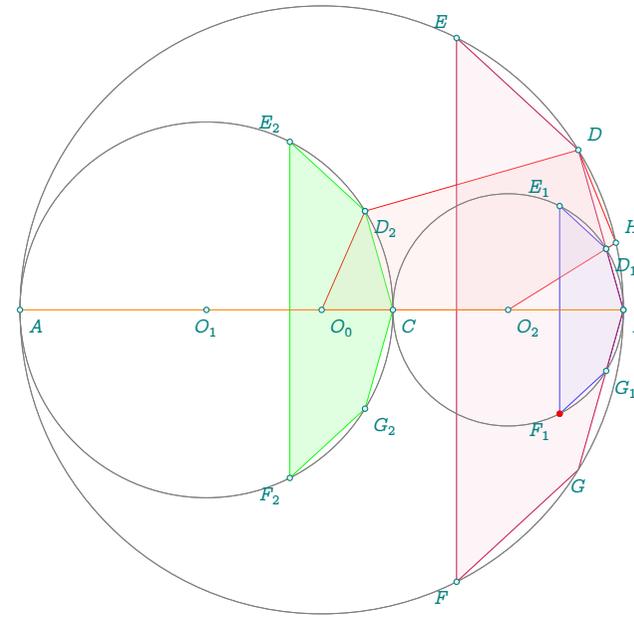
6.8 Golden Midcircles

$H_0 = M_2\,;\, E = O_2\,;\, F = O_0$

```
\tkzDivHarmonic[int](A,C,M_0)        \tkzGetPoint{M_1}
\tkzDivHarmonic[int](A,B,M_0)        \tkzGetPoint{M_2}
```
instead of
```
\tkzDivHarmonic[int](O_0,O_1,A)      \tkzGetPoint{M_1}
\tkzDivHarmonic[int](O_0,O_2,B)      \tkzGetPoint{M_2}
```
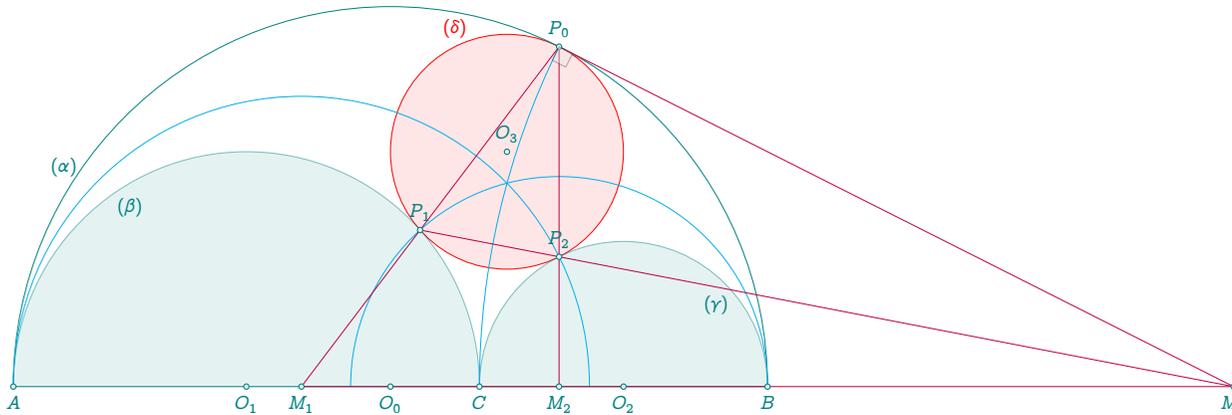

Figure 42: Golden Midcircles

We know $AM_1 = \dfrac{2a(a+b)}{2a+b}$.

With **golden arbelos**

$$AM_1 = \frac{2b\varphi(b\varphi + b)}{2b\varphi + b} = \frac{2b(\varphi^2 + \varphi)}{2\varphi + 1} = \frac{2b\varphi(\varphi + 1)}{\varphi + \varphi^2} = 2b$$

With **golden arbelos**, $BM_0 = \dfrac{2b^2}{b\varphi - b} = \dfrac{2b}{\varphi - 1} = 2b\varphi = 2a$

We deduce that $AM_0 = 2(a+b) + BM_0 = 2(a+b) + \dfrac{2b^2}{a-b} = \dfrac{2a^2}{a-b}$

Then $AM_0 = 4a + 2b$

On the one hand $M_0B \times M_0A = \dfrac{4a^2b^2}{(a-b)^2}$, on the other hand

$$M_0C^2 = (2b + M_0B)^2 = \left(2b + \frac{2b^2}{a-b}\right)^2 = \left(\frac{2b(a-b) + 2b^2}{a-b}\right)^2 = \frac{4a^2b^2}{(a-b)^2}$$

$M_0C = 2a + 2b = AB$ and $M_0C^2 = 4(a+b)^2$

6.8.1 Seventh circle in golden arbelos

It is easy to establish that $\dfrac{AD}{AR} = \dfrac{AR}{RD} = \phi$ and $\dfrac{AD}{AR} = \dfrac{AO_1}{AO_2} = \dfrac{a+b}{a} = \phi$

We know that $CD = 2\sqrt{ab}$. In the same way $\dfrac{BD}{SD} = \phi$

Using similar triangles and the Pythagorean theorem, it is easy to establish that $AD^2 = 4(a^2 + ab)$ and that $BD^2 = 4(b^2 + ab)$. It follows that $\dfrac{AD^2}{BD^2} = \dfrac{a^2 + ab}{b^2 + ab}$

or even that $\dfrac{AD^2}{BD^2} = \dfrac{\varphi^2 + \varphi}{1 + \varphi} = \varphi$. We deduce that $\dfrac{AD}{BD} = \sqrt{\varphi}$. In the same way $\dfrac{RC}{RD} = \sqrt{\varphi}$
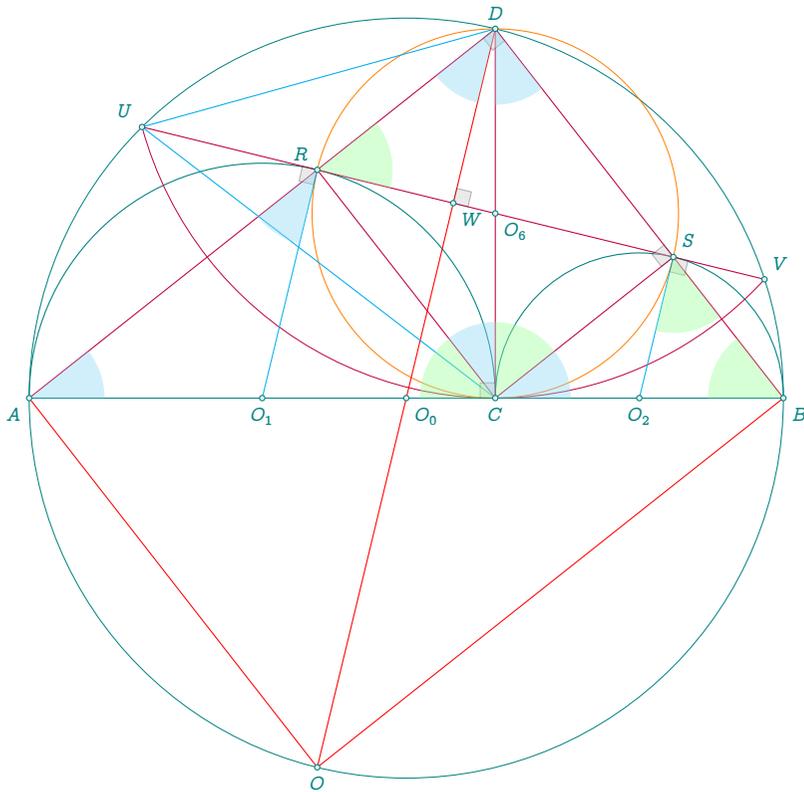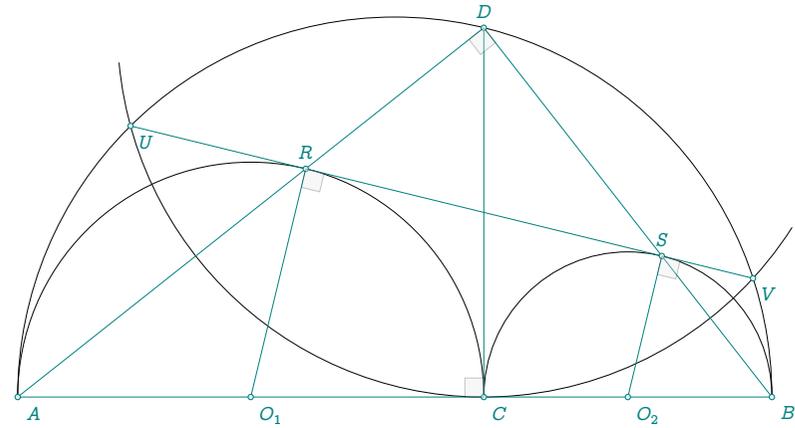
6.8.2 First consequences

Figure 44: Consequences

It is quite easy with the help of similar right-angled triangles to establish :
$BD = AC = 2a$, $CD = 2\sqrt{ab} = 2b\sqrt{\varphi}$ and $AD = 2a\sqrt{\varphi}$
$\dfrac{AD}{AR} = \dfrac{BD}{DS} = \varphi$

Figure 43: Seventh circles in golden arbelos

7 Lemmas

7.1 Menelaus' theorem

If line (PR) intersecting (AB) on triangle ABC, where P is on the extension of $[AB]$, R is on (BC), and Q on the intersection of (PR) and (AC), then
$$\frac{PA}{PB} \cdot \frac{QA}{QC} \cdot \frac{RC}{RB} = 1.$$
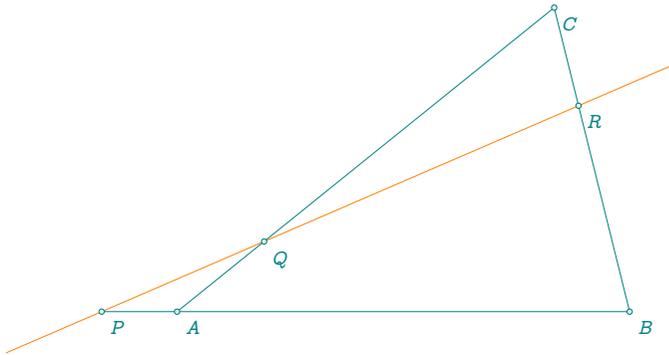Proof with Similar Triangles: draw a line parallel ...

Figure 45: Menelaus' theorem

The converse of Menelaus' theorem is also true. If $\dfrac{PA}{PB} \cdot \dfrac{QA}{QC} \cdot \dfrac{RC}{RB} = 1$ in the above diagram, then P, Q, R are collinear. The converse is useful in proving that three points are collinear.

7.2 Alternate segment theorem

The alternate segment theorem is the angle that lies between a tangent and a chord is equal to the angle subtended by the same chord in the alternate segment.

Converse of Theorem: If a line is drawn at one end of a chord of a circle in such a way that angles made with the chord are equal to alternate angles made by the chord in the segment then this line is the tangent line to the circle.
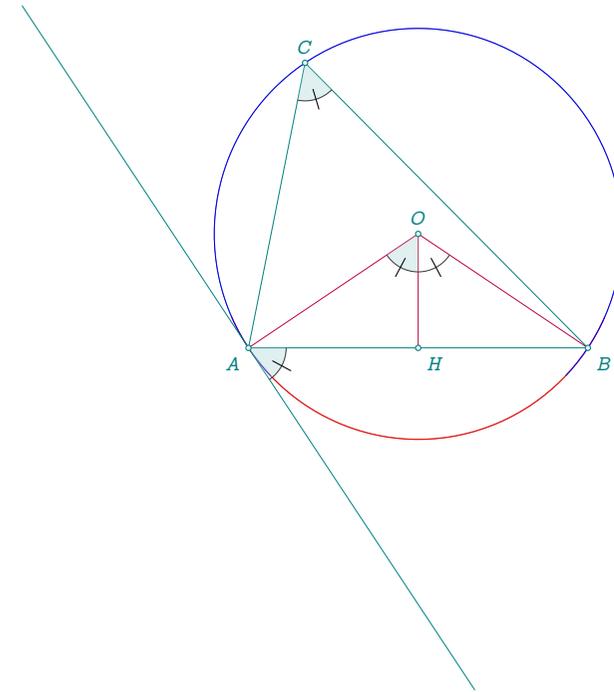

Figure 46: Alternate segment theorem

7.3 Angle Bisector Theorem

7.3.1 Internal angle bisector

In a triangle, the angle bisector of any angle will divide the opposite side in the ratio of the sides containing the angle.

Let E be the intersection of (AD) and the line parallel to (AB) through C. $\widehat{AEC} = \widehat{BAE}$, meaning that ACE is isosceles and thus $AC = CE$. The triangles ABD and ECD are similar, implying
$$\frac{AB}{CE} = \frac{BD}{CD}$$
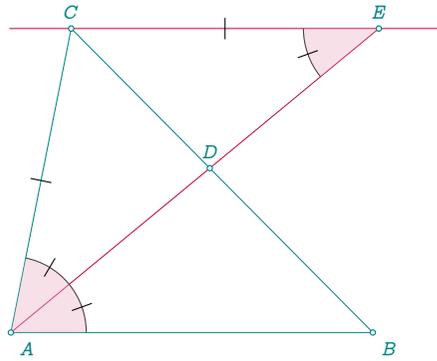With $CE = AC$, we obtain $\dfrac{AB}{AC} = \dfrac{BD}{CD}$.

Figure 47: Internal angle bisector

7.4 Harmonic division

Let A, C, B, and D four points which lie in this order on a line d. The four-point $(A, B; C, D) = \dfrac{CA}{CB} \div \dfrac{DA}{DB}$ is called a harmonic division if $(A, B; C, D) = 1$

C and D are harmonic conjugates with respect to A and B.

If X is a point not lying on d, then we say that pencil which consists of the four lines (XA), (XB), (XC), (XD) is harmonic if $(A, B; C, D)$ is harmonic.

7.3.2 External angle bisector

The external bisector of an angle $(\widehat{ACB})$ of a triangle (ABC) divides the opposite side externally in the ratio of the corresponding sides containing the angle.
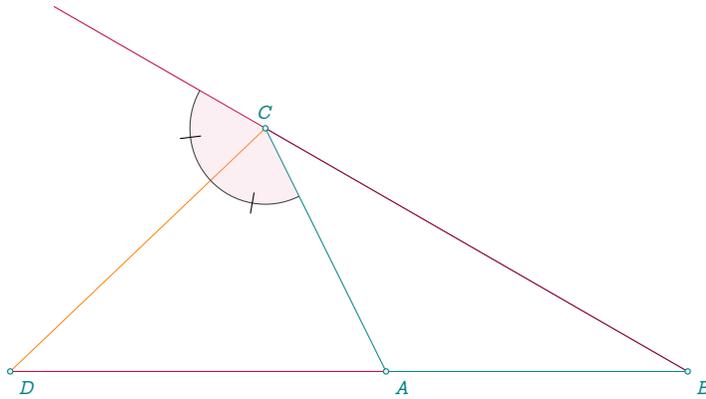

Figure 48: External angle bisector

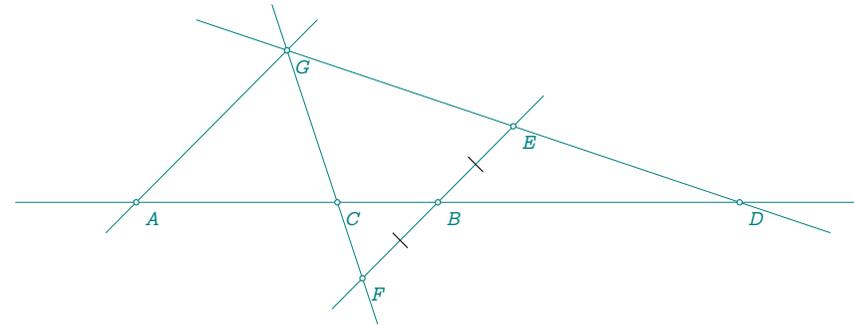$$\frac{DA}{DB} = \frac{CA}{CB}$$

Figure 49: Harmonic division v1

How to get the harmonic conjugate C of a point D outside the segment $[AB]$.

We choose a point X not aligned with A and B. Here I choose X such that (AX) is orthogonal to (AB) and with $AB = AX$. Then I denote by E the middle of $[XB]$.

F is the intersection of (DE) with (XA), then G is the intersection of (AE) with (BF) and finally C is the intersection of lines (XG) and (AB)
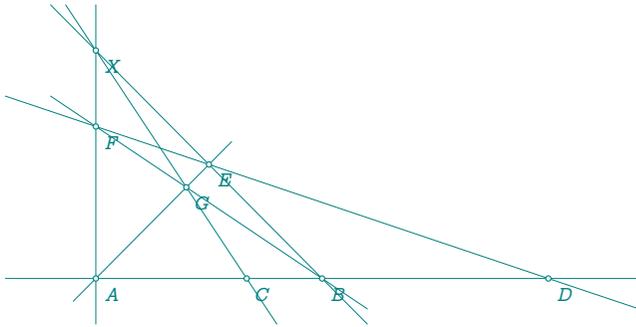
Figure 50: Harmonic division v2

7.5 Bisector and harmonic division

Let four points A, C, B and D, in this order, lying on the straight line (d) and M un point pris hors de (d). Then, if two of the following three propositions are true, then the third is also true:
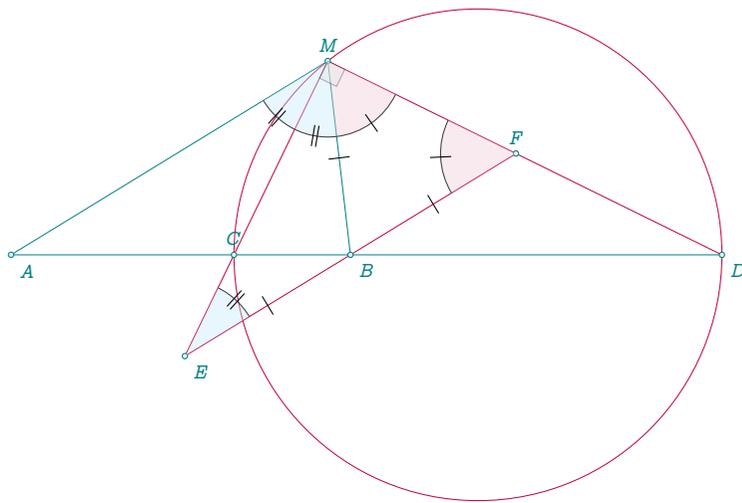

Figure 51: Bisector and harmonic division

1. The division (A,B;C,D) is harmonic. $(CA/CB = DA/DB)$

2. (MC) is the internal angle bisector of $\widehat{AMB}$.

3. $(MD) \perp (MC)$.

Proof: (2) and (3) impliquent (1)

C point of intersection of the internal bisector of $\widehat{AMB}$, D point of intersection of the external bisector of $\widehat{AMB}$.

We draw the parallel to (AM) passing through B. It cuts (CM) at E and (MD) at F.

On one hand $\dfrac{CA}{CB} = \dfrac{AM}{BE} = \dfrac{MA}{MB}$

on the other hand $\dfrac{DA}{DB} = \dfrac{AM}{BF} = \dfrac{MA}{MB}$

It follows: $\dfrac{CA}{CB} = \dfrac{DA}{DB} = \dfrac{MA}{MB}$

We conclude that $(A,B;C,D)$ is harmonic.

and the lines (MA), (MB), (MC) and (MD) form a harmonic pencil.

The circle of diameter $[CD]$ is the circle of Apollonius for the triplet (A,B,k) with $k = \dfrac{CA}{CB}$.

7.6 Apollonius circle

> **Apollonius circle**
>
> *Apollonius of Perga proposed the following problem: "Find the locus of a point the ratio of whose from two fixed points is constant". Let $k \neq 1$ be the ratio such that $\dfrac{MA}{MB} = k$.*
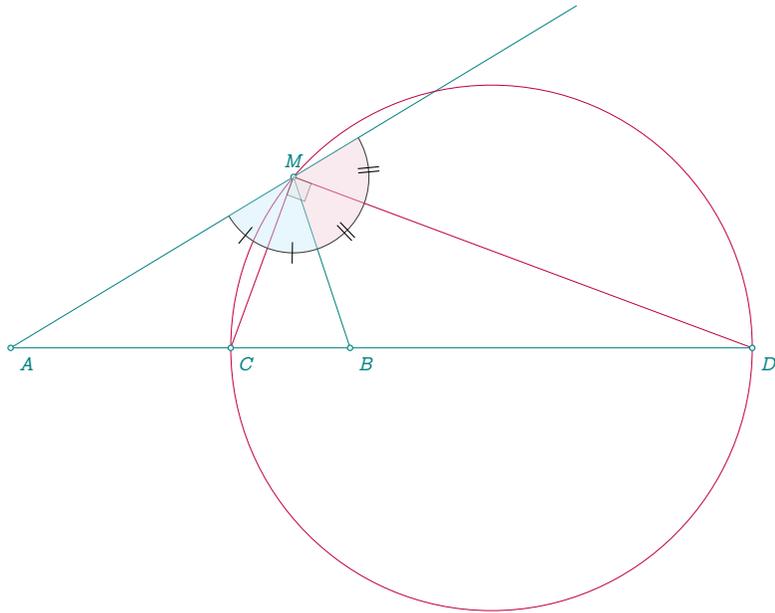
Figure 52: Apollonius circle

Figure 53: Orthogonal circles

Consider the point C on the segment $[AB]$ satisfying the ratio $\dfrac{CA}{CB} = \dfrac{MA}{MB} = k$.

From the "angle bisector theorem" $\widehat{AMC} = \widehat{CMB}$. Next take the point D on the line (AB) that satisfies the ratio. By the "angle bisector theorem" the line (MD) bisects the exterior angle $\widehat{EMB}$. It's easy to show that (CM) and (DM) are orthogonal in M. So M is on the circle with diameter $[CD]$.

7.7 Orthogonal circles

> **Orthogonal circles**
>
> *In geometry, two intersecting circles in a plane are said to be orthogonal if at each of the two points of intersection the tangents to both circles are orthogonal. By reason of symmetry, it is sufficient that the previous property occurs at one of the points of intersection.*
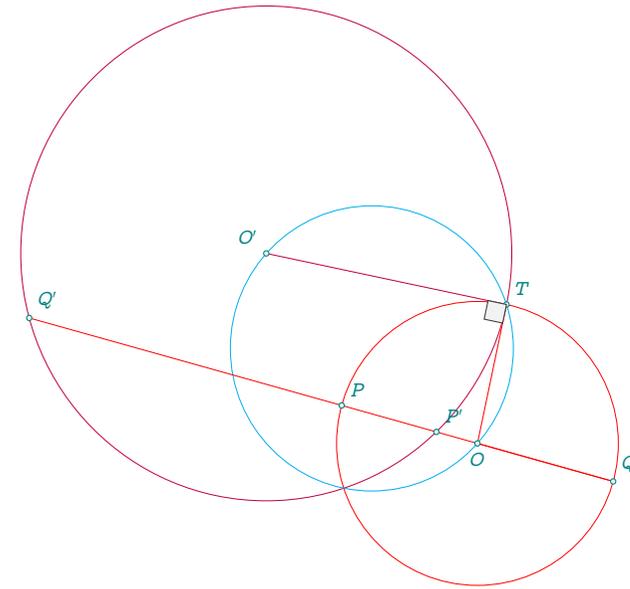
For two circles to be orthogonal, it is necessary and sufficient that there is a diameter of one of them that is harmonically divided by the other. Indeed the power of the point O with respect to the circle (O) is $OT^2 = OP' \times OQ'$. Thus $OP^2 = OQ^2 = OP' \times OQ'$. $[P, Q, P', Q']$ is a harmonic division according to Newton's relation.

Application : given a circle (O) and a point M, distinct from the center O and not belonging to the circle, to find the circles orthogonal to (O) passing through M, draw the diameter $[PQ]$ on the line (OM) and find the point N such that $[P, Q, M, N]$ is a harmonic division: Any circle passing through M and N, centered on the midpoint of $[MN]$, is orthogonal to (P). The set of circles passing through M and orthogonal to (O) is a pencil of circles with base points M and N.

see the pencils

7.8 Apollonius circles and circumcircle of a triangle

> **Apollonius circles and circumcircle of a triangle**
>
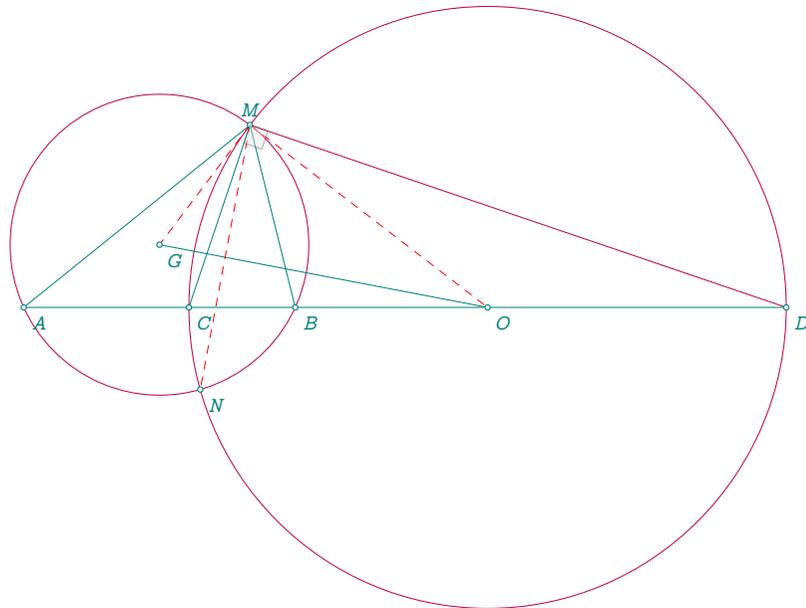> *The apollonius circles of triangle are orthogonal to the circumcircle of this triangle.*

Figure 54: Apollonius circles and circumcircle of a triangle

$\widehat{OMB} + \widehat{BMC} = \widehat{OMC} = \widehat{OCM} = \widehat{CAM} + \widehat{AMC}$
or $\widehat{AMC} = \widehat{BMC}$ so $\widehat{OMB} = \widehat{CAM} = \widehat{BAM}$
Hence by the alternate segment theorem the result follows. (OM) is tangent at M to the circumcircle of the triangle ABC.
Another solution is to prove that $OM^2 = OB \cdot OA$ i.e. A and B are inverses with respect to the circle with diameter $[CD]$.
From $\dfrac{CA}{CB} = \dfrac{DA}{DB}$ and $OM = R$
$CB = R - OB, DB = R + OB, AD = R + OA$ and $AC = OA - R$
then $(OA - R)(OB + R) = (R - OB)(OA + R)$.

Finally after simplification

$$R^2 = OM^2 = OA.OB$$

7.9 Apollonius circles of a triangle

> **Apollonius circles of a triangle**
>
> *The three Apollonius circles of a (non-equilateral) triangle meet at exactly two points.*

Proof:

Let ABC be a non-equilateral triangle, say with $AB \neq AC$. By definition, we have directly that if a point belongs to two of the circles, then it belongs to the third. Indeed, if

$$\frac{MB}{MC} = \frac{AB}{AC} \text{ and } \frac{MA}{MB} = \frac{CA}{CB} \text{ then } \frac{MA}{MC} = \frac{BA}{BC}$$

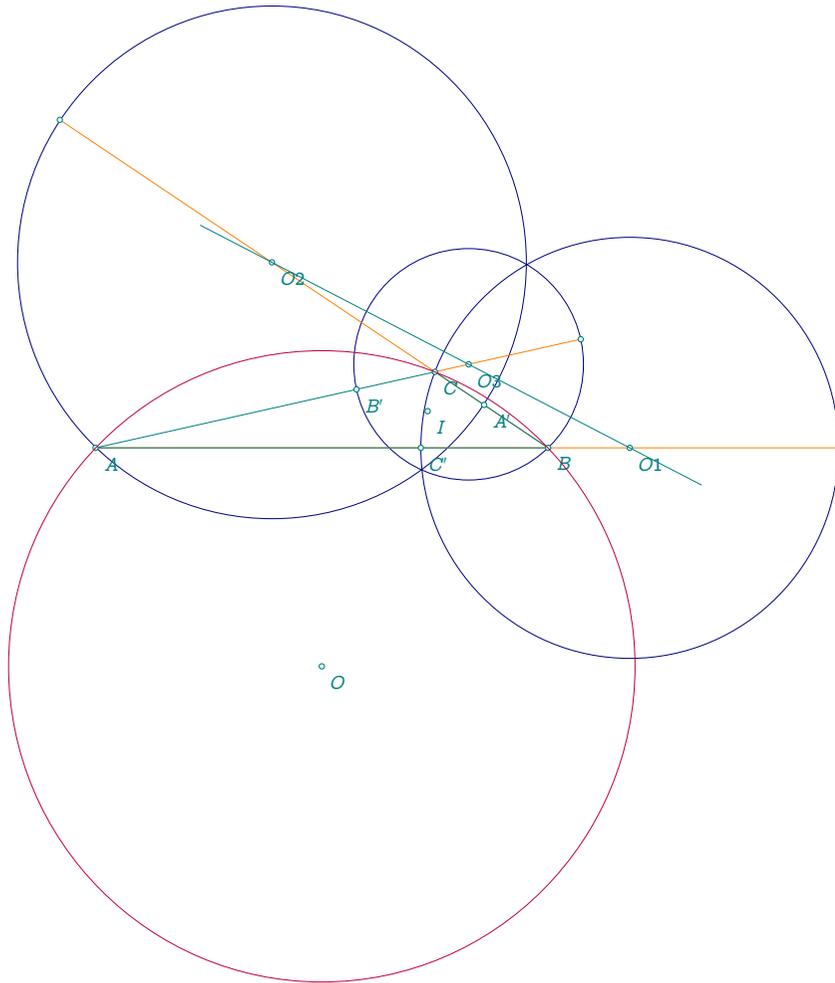However, the circles are neither tangent nor disjoint if $AB \neq AC$.

$\dfrac{O1B}{O1C} = \dfrac{BC}{AC} = \dfrac{a}{b} = \dfrac{O1C}{O1A}$ with $AB = c\ BC = a\ AC = b$. Thus $\dfrac{O1B^2}{O1C^2} = \dfrac{a^2}{b^2}$. But from power of the point $O1$ we have $O1C^2 = O1A \cdot O1B$ (This equality can also be obtained with relations in similar triangles).

This leads to $\dfrac{O1B}{O1A} = \dfrac{a^2}{b^2}$. Also $\dfrac{O1A}{O1C} = \dfrac{c^2}{a^2}$ and $\dfrac{O1C}{O1B} = \dfrac{b^2}{c^2}$.

Multiplying the three expressions, from the converse of the Menelaus's theorem we conclude that $O1, O2$ and $O3$ are aligned and the three Apollonius circles are coaxal.

7.10 Power of a point with respect to a circle

> **Power of a point with respect to a circle**
>
> *The power of a point M with respect to a circle C is the number $k = \overline{MA} \cdot \overline{MB}$ where A and B are the points common to C and any line containing M. These points are of course merged if the line is tangent to the circle. The number k is positive if the point M is outside the circle, null if it belongs to it and negative if it belongs to the open disk defined by the circle. We can easily verify that $k = d^2 - R^2$ where d is the distance from point M to the center of the circle and R is the radius of the circle. k est aussi noté $C(M)$. On a donc $C(M) = \overline{MA} \cdot \overline{MB} = d^2 - R^2$.*

Figure 55: Apollonius circles of a triangle

> **Apollonius circles are coaxal**
>
> *The three Apollonius circles are coaxal. Their centers are aligned.*

$(O1C)$ is a tangent line to the circumscribed circle of triangle ABC. We deduce that the triangles $O1BC$ and $O1AC$ are similar which leads to
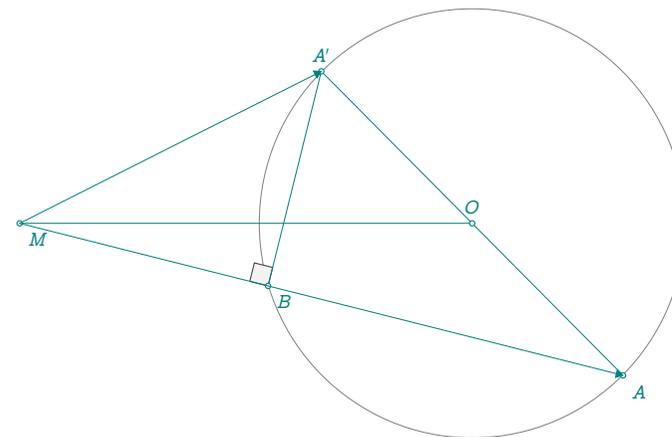

Figure 56: Power of a point with respect to a circle

7.10.1 Properties of the power

1. The power of a point M with respect to a circle $\mathcal{C}$ of diameter AB is equal to the scalar product $\overrightarrow{MA} \cdot \overrightarrow{MB}$

   $$\mathcal{C}(M) = \overrightarrow{MA} \cdot \overrightarrow{MB}.$$

2. If we draw two secants (MAB) and (MCD) to the circle $\mathcal{C}$ we obtain :

   $$\mathcal{C}(M) = \overrightarrow{MA} \cdot \overrightarrow{MB} = \overrightarrow{MC} \cdot \overrightarrow{MD}$$

   If the point M is outside the circle, denoting by MT a tangent :

   $$\mathcal{C}(M) = MT^2 = \overline{MA} \cdot \overline{MB}$$

   If the point M is interior , designating by $[UV]$ the perpendicular chord to (OM):

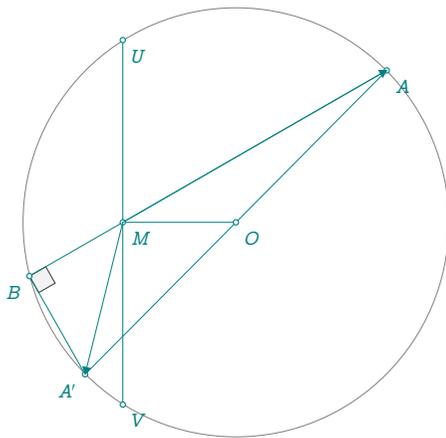 $$\mathcal{C}(M) = -\overline{MU}^2$$

Figure 57: Properties of the power

3. For the power $\mathcal{C}(M)$ to be positive, zero or negative, it is necessary and sufficient that d is greater than, equal to or less than R, so that M is outside, on the circle or inside the circle $\mathcal{C}$. The minimum of $\mathcal{C}(M)$ is reached when M is in $O : \mathcal{C}(M) = -R^2$.

4. If the sides AB and CD of the quadrangle $ABCD$ intersect at a point M such that $\overrightarrow{MA} \cdot \overrightarrow{MB} = \overrightarrow{MC} \cdot \overrightarrow{MD}$, this quadrangle is inscribable.

5. If the point M on side AB of triangle ABC is such that $MC^2 = MA \cdot MB$ the line (MC) is tangent at C to the circle ABC.

7.11 Radical axis

> **Radical axis**
>
> *The geometric set of points that have the same power with respect to two given circles is a line perpendicular to the line of the centers, called radical axis of the two circles. If the circles have two points in common, the radical axis is the common secant line of the circles.*

For the point M to have the same power with respect to the two circles $\Gamma = O(R)$ and $\Gamma' = O'(R')$, it is necessary and sufficient that :

$$MO^2 - R^2 = MO'^2 - R'^2 \text{ either } MO^2 - MO'^2 = R^2 - R'^2$$

Denoting by I the middle of OO', the set of points M is therefore a line OO' perpendicular to the line OO' at the point H defined by the relation:

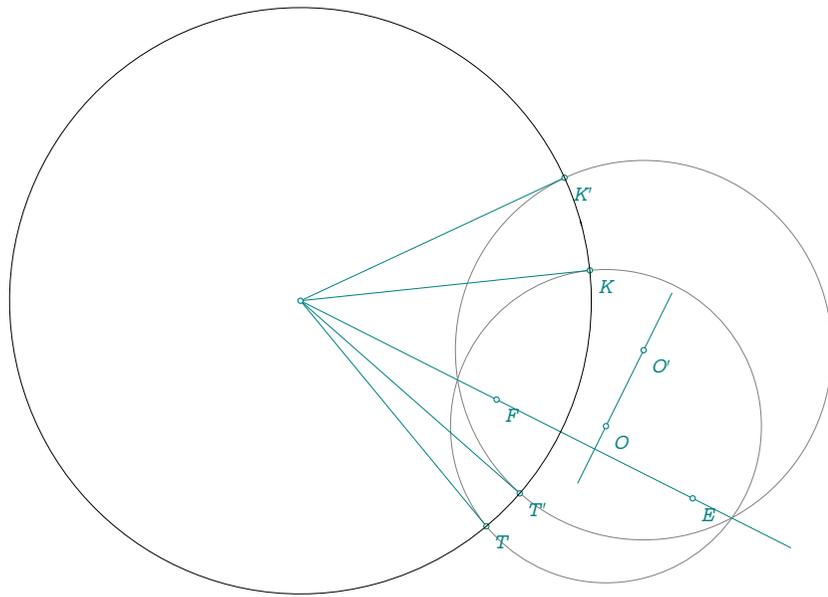$$2\overline{OO'} \cdot \overline{IH} = R^2 - R'^2$$
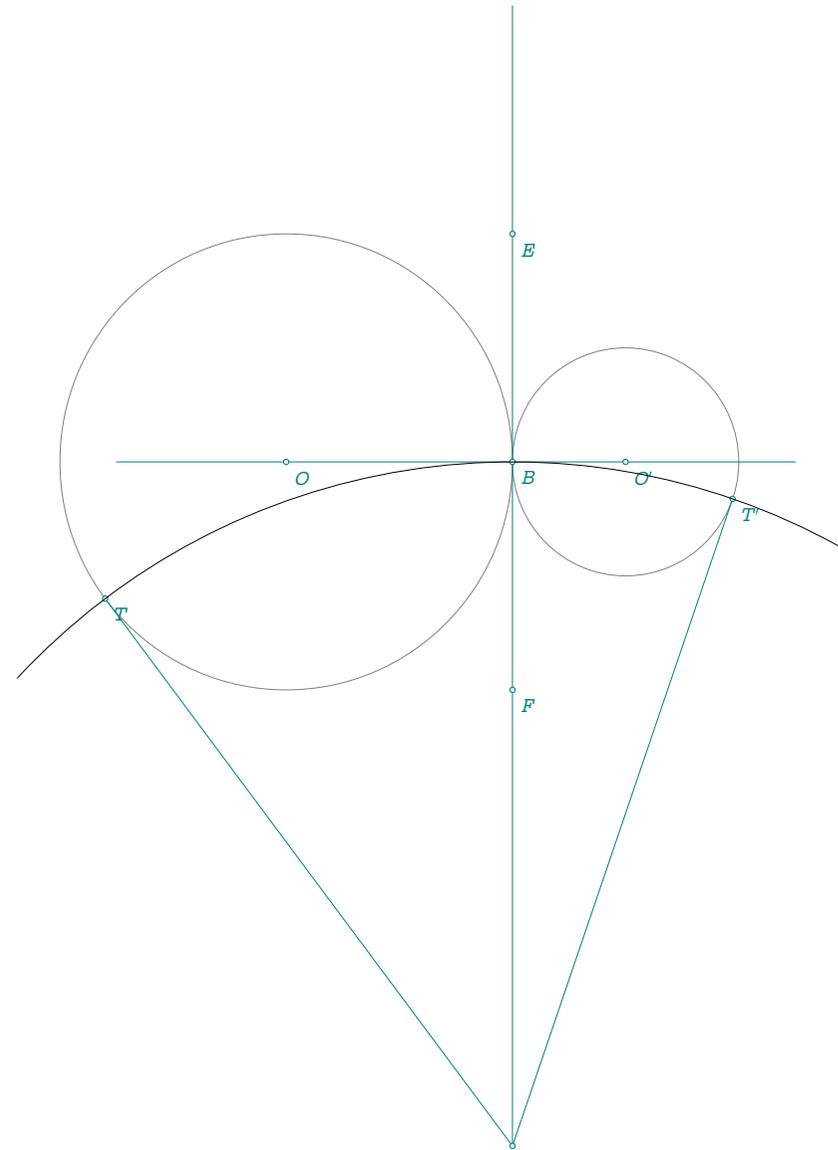
Figure 59: Radical axis 2
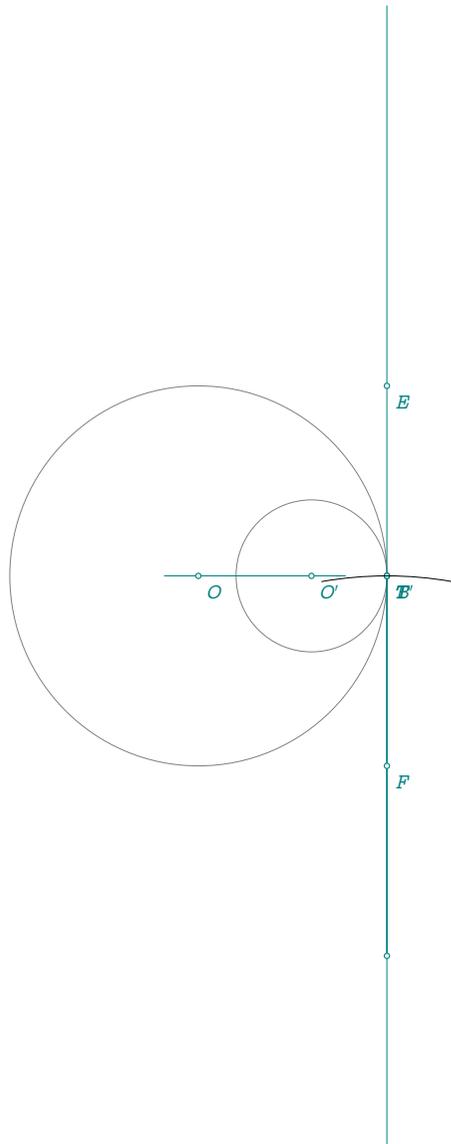

Figure 60: Radical axis 3
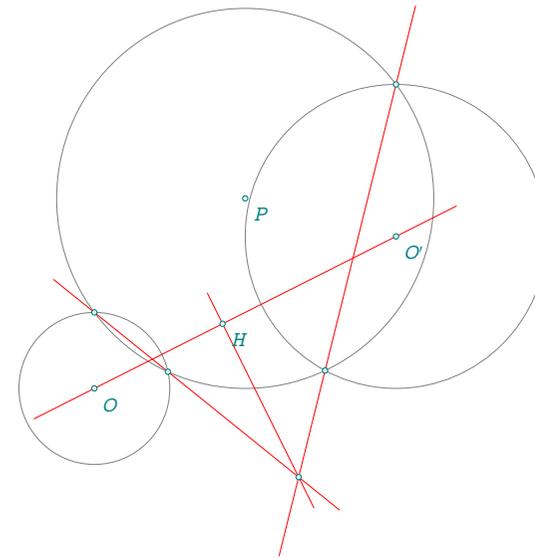
Figure 61: Radical axis 4

Figure 62: Radical axis 5

7.12 Construction of the radical axis for two non-intersecting circles

The last construction shows that in a general case the radical axes of three circles intersect at a single point. This construction allows us to construct the radical axis of two non-intersecting circles. Let us draw an auxiliary circle that intersects the first circle at A and B and the second at A' and B'. The lines (AB) and $(A'B')$ intersect at a point M which will belong to the radical axis. This one being perpendicular to the line of centers, it remains to construct the orthogonal projection of M on the line of centers.
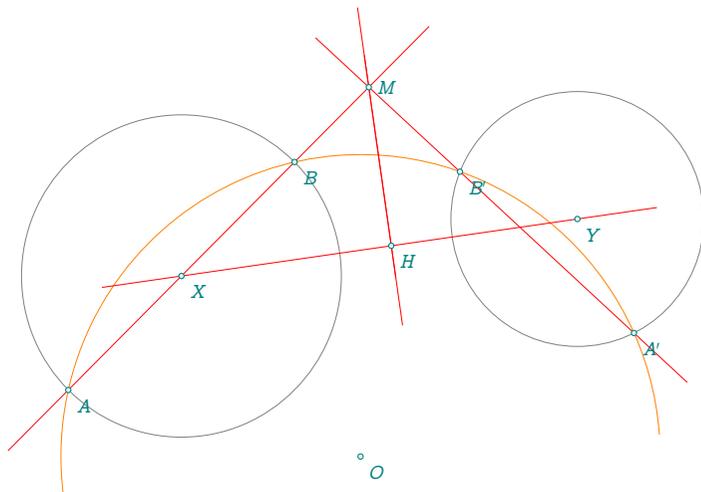
Figure 63: Radical axis construction

The part of the radical axis, outside the two circles, is the place of the points from which equal tangents can be drawn.

Indeed, for $MT = MT'$ it is necessary and sufficient that M has the same positive power with respect to the circles O and O'. It follows that the radical axis of of two circles passes through the middle of any line common to these two circles. When two circles are external, the middles of the 4 common tangents are aligned on the radical axis.

7.12.1 Powers of a point with respect to two circles

┌─ Powers of a point ──────────────────────────────────────┐
│   │
│ *The difference of the powers of a point with respect to two circles is, in absolute* │
│ *value, equal to the double product of the distance of their centers by the* │
│ *distance of this point to their radical axis.* │
│   │
└──┘

(see 7.11)

7.12.2 Circle orthogonal to two given circles

┌─ Circle orthogonal to two given circles ─────────────────────────────────────┐
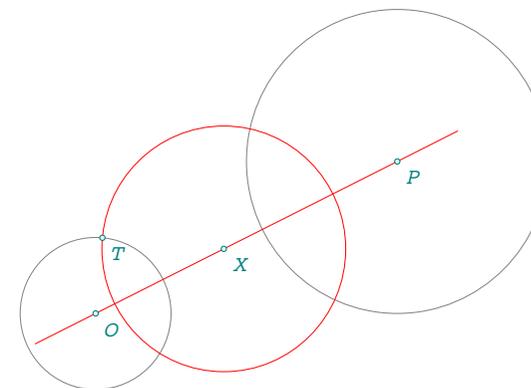│   │
│ *If a circle is orthogonal to two given circles, its center lies on the radical axis* │
│ *of those two circles.* │
│   │
└──┘

7.13 Radical center and radical circle

– Case of two circles

```
z.X      = C.Ox : radical_center(C.Pz)
C.X      = C.Ox : radical_circle (C.Pz)
```


– Case of three circles

```
C.ortho   = C.exa : radical_circle (C.exb,C.exc)
```

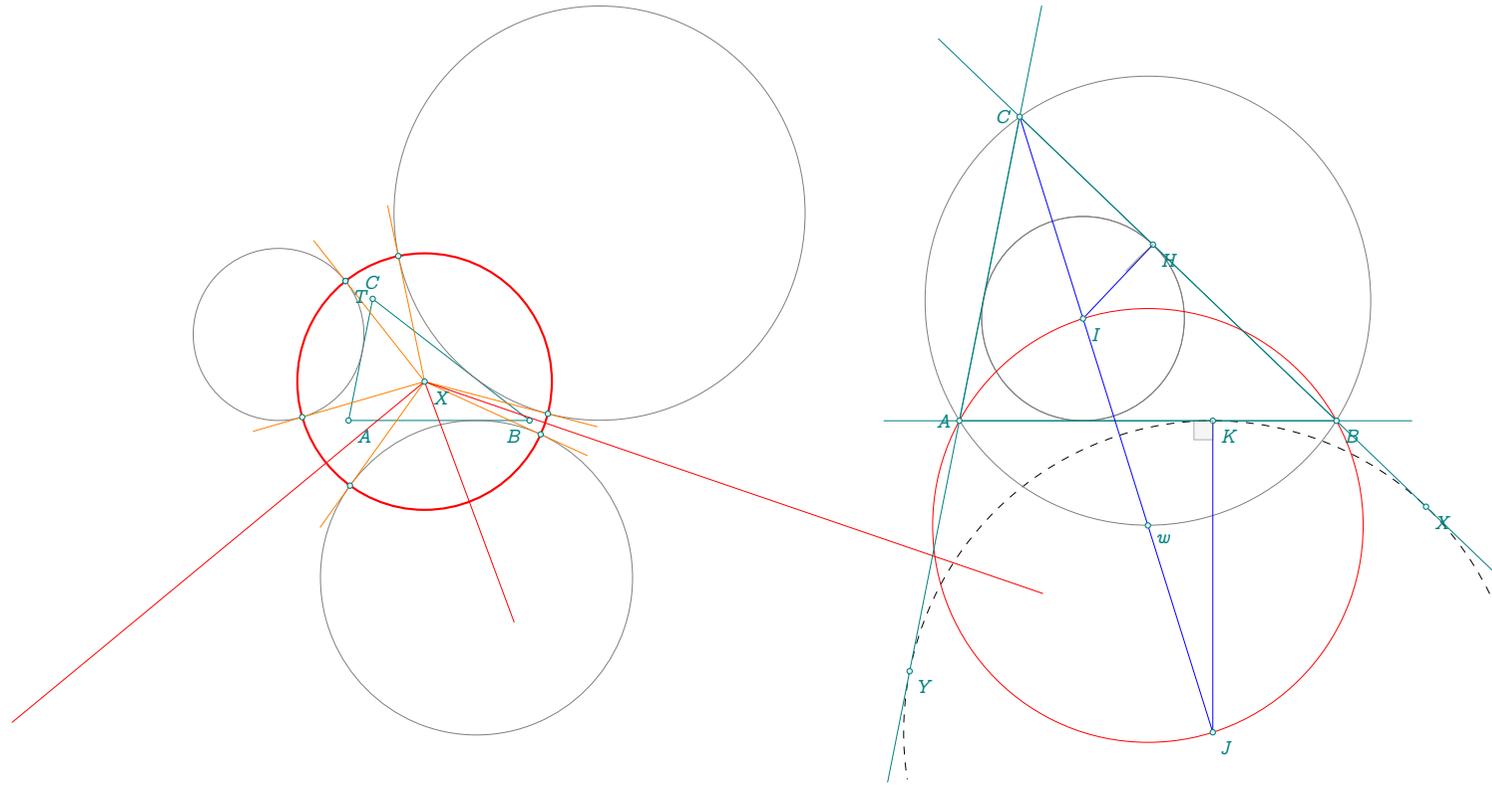The center is intersection of the three radical axis.

Figure 64: Euler's relation

The quadrilateral $AIBJ$ having two opposite right angles is inscribed in a circle whose center ω is the middle of IJ. This center ω belongs to the perpendicular bisector of $[AB]$ and to the interior bisector of the angle $\widehat{ACB}$ and is thus the middle of the arc $\widehat{AB}$ south pole of the circle circumscribing the triangle ABC. (7.22).

7.14 Euler's relation

Let (Γ) be the circumscribed circle of center O to the triangle ABC. Let I and J be the centers of the inscribed and exinscribed circles in the angle $\widehat{A}$ to the triangle ABC admitting $r1$ and $r2$ as respective radii

7.15 Pencil of circles

A pencil of circles or coaxial system is any family of circles which have the same radical axis Δ. Any two circles in the plane have a common radical axis, which is the line consisting of all the points that have the same power with respect to the two circles. This is the case, for example, for circles passing through two fixed points A and B or of circles orthogonal to two fixed circles. We will show that a

pencil is defined when we know a circle of this pencil and the common radical axis Δ and consequently when we know two circles of the pencil. The centers of the circles of a pencil are aligned on a perpendicular to the radical axis Δ of the pencil alled "line of centers of the pencil ".

7.15.1 Fixed point pencil

The fixed points A and B are the "base points" of the pencil.

The circles of the beam are the circles passing through the base points A and B.

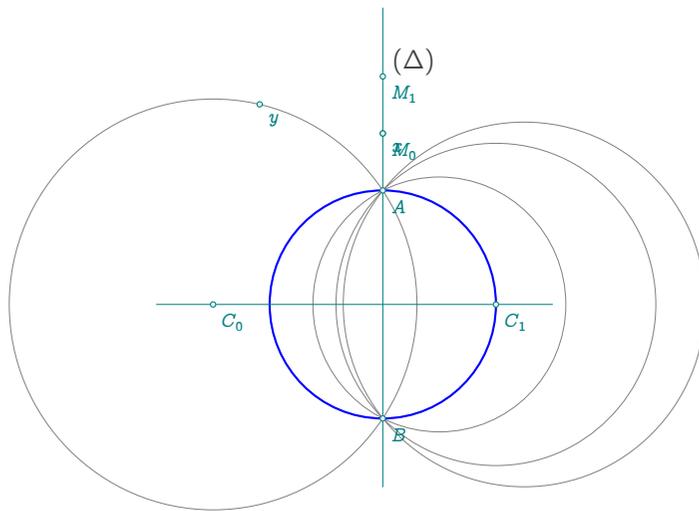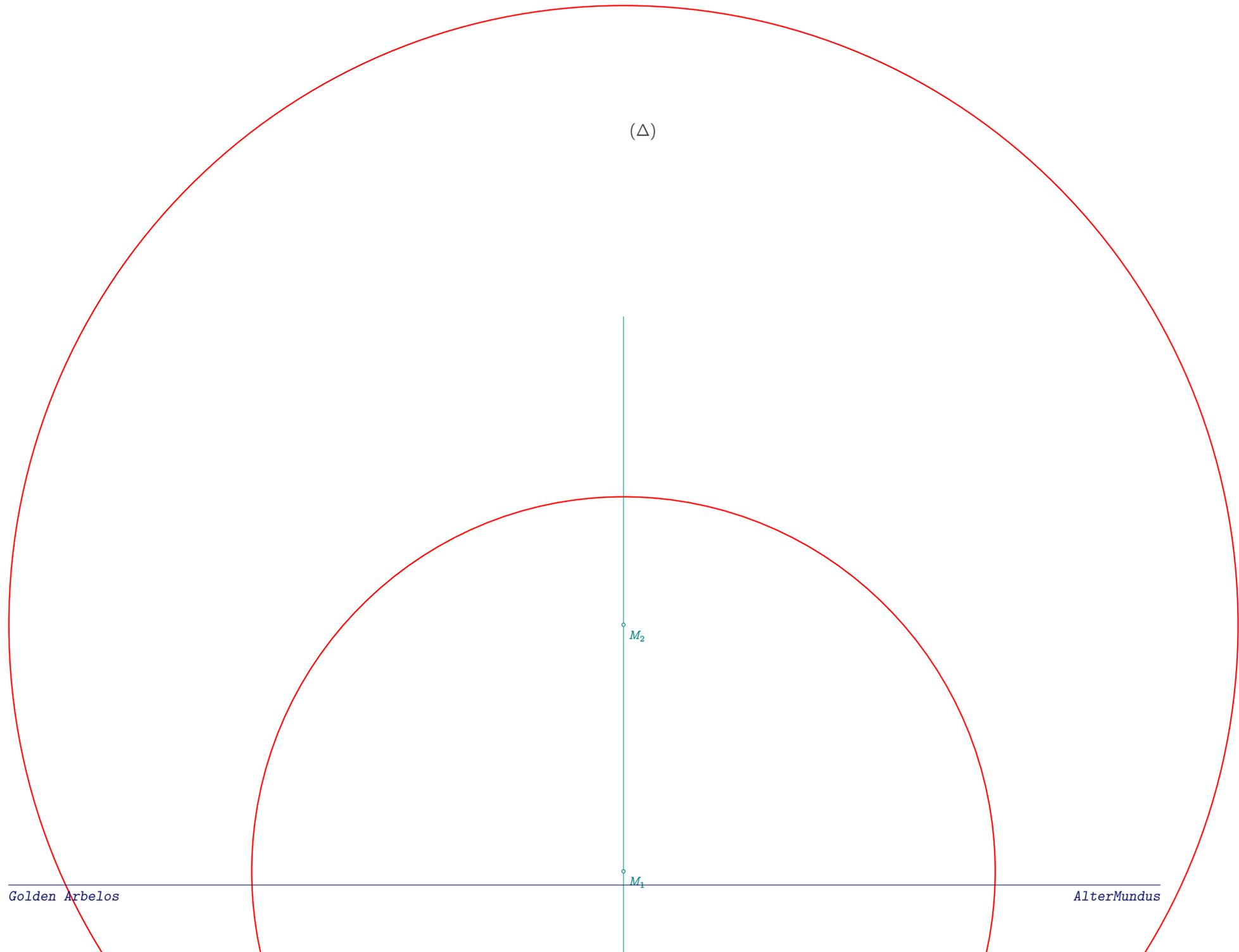The pencil has a smallest circle, having as diameter the segment $[AB]$.

Figure 65: Pencil of circles a

(Δ)

M_2

M_1

7.16 Cocyclic points

Cocyclic points

Let i be an inversion of pole w and power k, and $\mathcal{C}$ a circle containing w. Consider the point A of $\mathcal{C}$ diametrically opposite w and its inverse H by i. Let M be any point of $\mathcal{C}$ distinct from A and from w, of inverse M'. The equality $\overline{wA} \cdot \overline{wH} = \overline{wM} \cdot \overline{wM'}$ (each member being equal to k) shows that the points A, H, M', M are cocyclic.

To convince oneself of this, consider, for example, the circle γ circumscribed by the triangle AHM and the power of w with respect to this circle : $\overline{wA} \cdot \overline{wH} = \overline{wM} \cdot \overline{wN}$ where N is the other point common to the circle γ with the line (wM). Then deduce that N is none other than M'. This being the case, since the angle $\widehat{AMM'}$ is right, so is the angle $\widehat{AHM'}$. Also, when M describes $\mathcal{C}$, M' describes the line Δ perpendicular at H to the line (wA). The image by i of the circle $\mathcal{C}$ (which contains the pole w) is the line Δ.
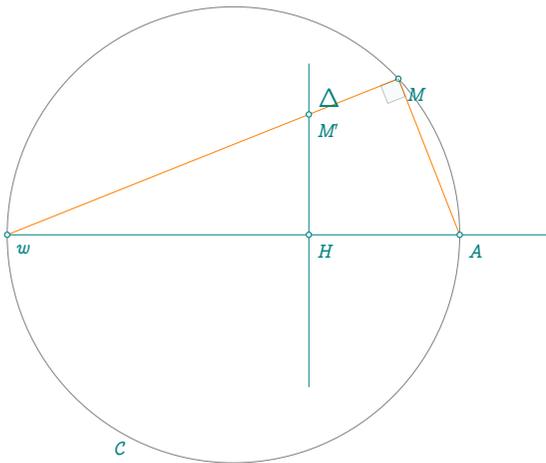

Figure 67: Cocyclic points

7.17 Archimedes' Book of Lemmas

7.17.1 Proposition 1

Proposition 1

If two circles touch at A, and if $[CD], [EF]$ be parallel diameters in them, the points $A, C,$ and E are aligned.
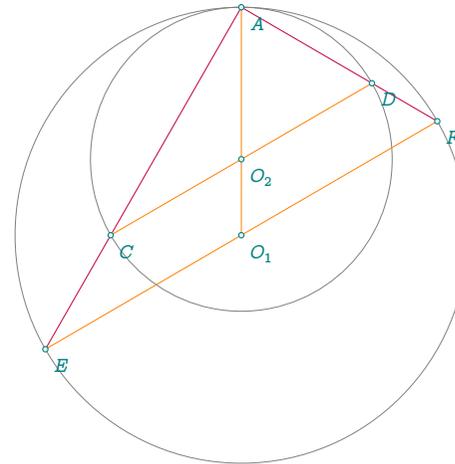

Figure 68: Archimedes' Book of Lemmas prop. 1

$(CD) \parallel (EF)$

(AO_0) is secant to these two lines so $\widehat{A0_2C} = \widehat{A0_1E}$.

Since the triangles AO_1C and AO_0E are isosceles the angles at the base are equal $\widehat{AC0_2} = \widehat{AE0_1} = \widehat{CA0_2} = \widehat{EA0_1}$. Thus A, C and E are aligned.

7.18 D'Alembert's Theorem

Figure 70: D'Alembert's theorem v2

> **D'Alembert's theorem**
>
> *If three circles A, B, and C are taken in pairs, the external similarity points of the three pairs lie on a straight line (I, J, K are aligned). Similarly, the external similarity point of one pair and the two internal similarity points of the other two pairs lie upon a straight line, forming a similarity axis of the three circles (Two inner centers are aligned with an outer center: I', K', J or I', J', K or I, J', K').*
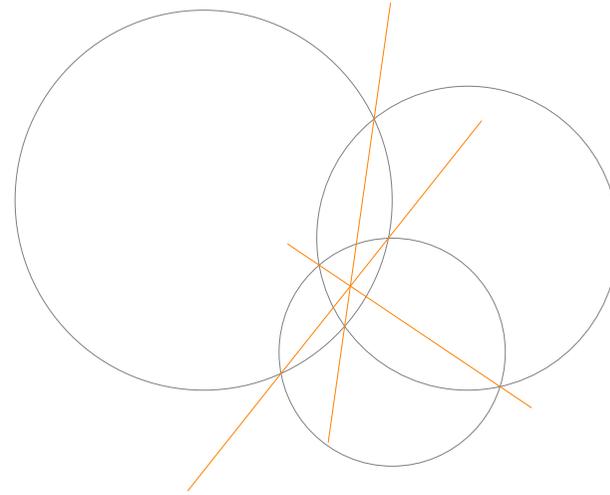
7.19 Altshiller-Court's theorem
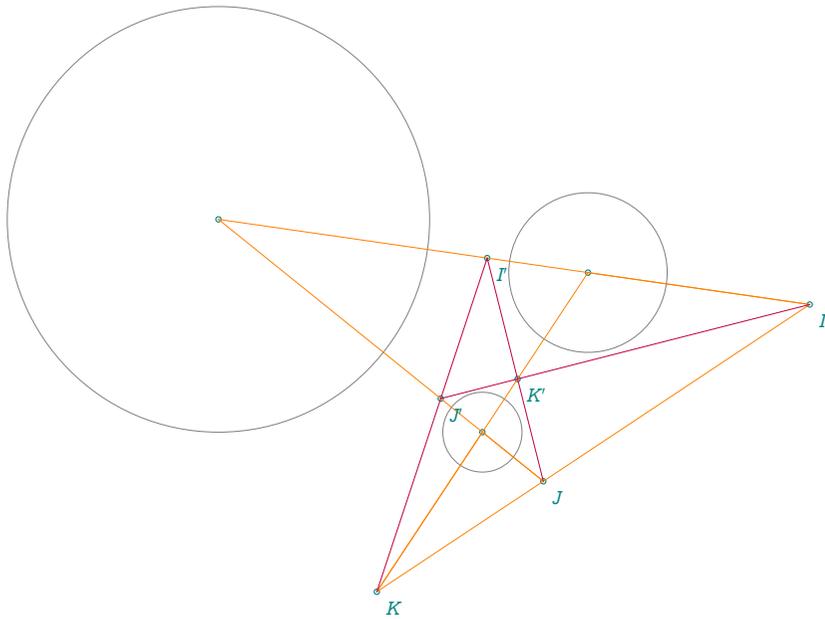
> **Altshiller-Court's theorem**
>
> *The two lines joining the points of intersection of two orthogonal circles to a point on one of the circles met the other circle in two diametricaly oposite points. Altshiller p 176*
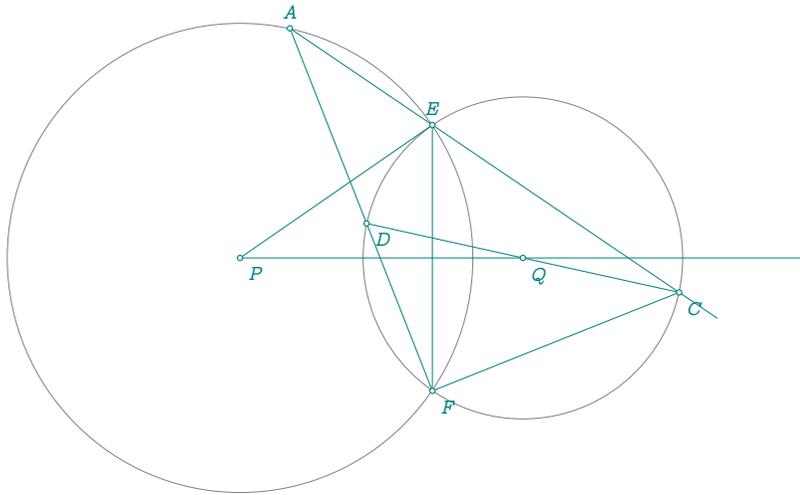

Figure 69: D'Alembert's theorem v1

Figure 71: Altshiller-Court's theorem 372 p 176

7.20 Reim's theorem

> **Reim's theorem**
>
> *Given two circles of center A and B intersecting at points C and D, and E and F any two points chosen on one of the circles, the lines (EC) and (FD) intersect the second circle at points H and G. We show that the lines (EF) and (HG) are parallel.*
>
> *Converse - Suppose we have two circles ω_1, ω_2 that intersect at two points C, D. Let the line through C intersects the two circles at E, H (ω_1, ω_2 respectively). Then suppose a line that cuts ω_1 at F and ω_2 at G in a way that $EF \parallel HG$. Then that line passes through D.*

Proof -

Proof: Let the lines $(EA), (FA)$ joining the points of intersection E, F of the two orthogonal circles $(P), (Q)$ to the point A of (P) met (Q) again in C, D (Fig.7.19). We have:

$$\widehat{EAF} = \frac{1}{2}\widehat{EPF} = \widehat{EPQ} \text{ and } \widehat{ECF} = \frac{1}{2}\widehat{EQF} = \widehat{EQP}$$

Hence $\widehat{EAF} + \widehat{ECF} = \widehat{EPQ} + \widehat{EQP} = 90°$ and therefore the triangle CAF is right-angled at F. Thus CD subtends a right angle at F,which proves the proposition.

(EF) and (CD) are antiparallel with respect to the sides of an angle O. (CD) and (GH) are antiparallel with respect to the sides of an angle O. (EF) and (GH) are parallel. (Furthermore, if (PQ) and (RS) are antiparallel, then the points P, Q, R, and S are concyclic)

Converse - Suppose that the line (FD) intersects the circle of center B at K. Through H there is only one parallel to (EF) so G, H, K are aligned. As H and K are two points of the circle they are coincident.
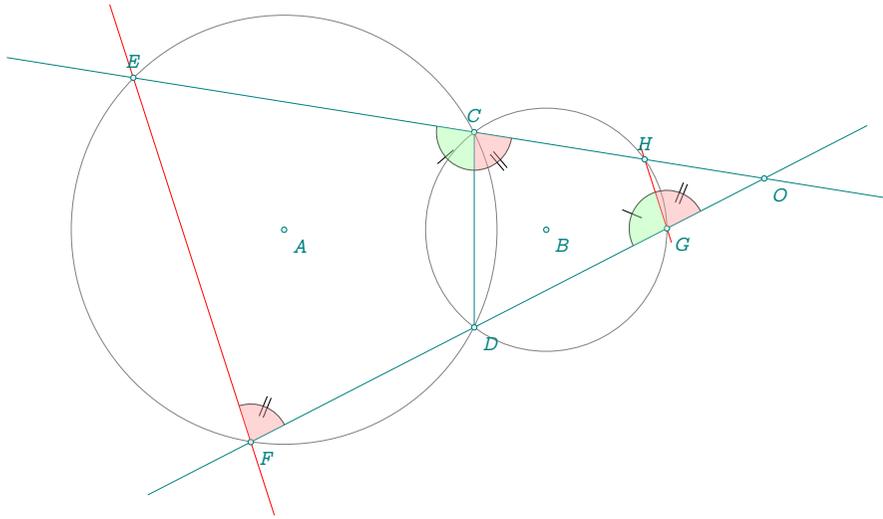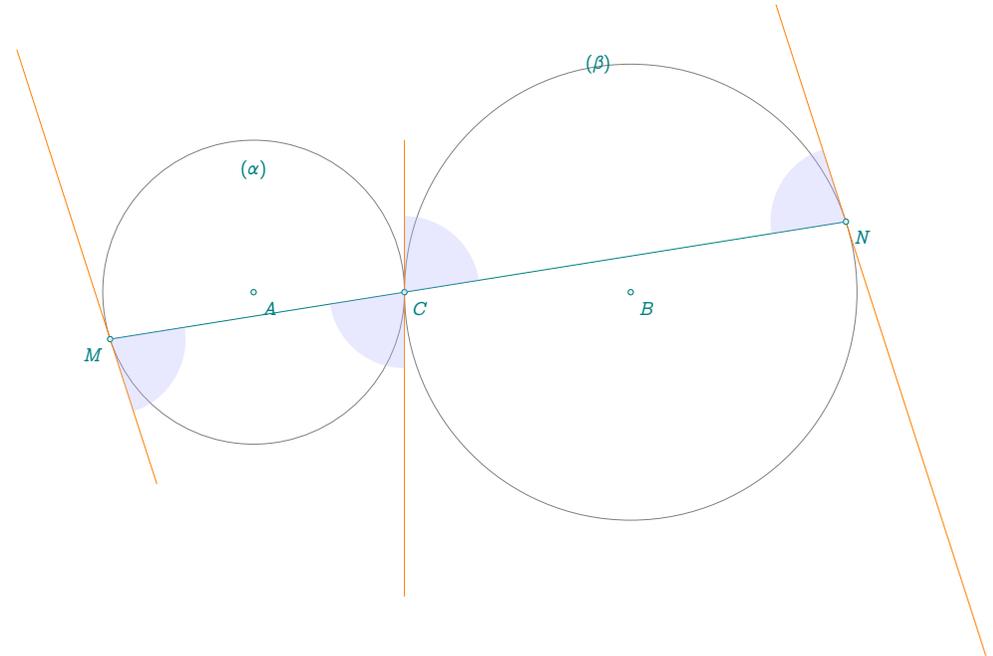
Figure 72: Reim

Figure 73: Reim 2a

7.20.1 Reim 2

Let two circles (α) and (β) be tangent at C. Let a line passing through C intersect the circle (α) at M and intersect (β) at N. The tangents at M and N respectively to the circles (α) and (β) are parallel.

Given a cyclic quadrilateral $ABQP$ and points P' and Q' on the extensions of $[PA]$ and $[FB]$, respectively. If $(PQ) \parallel (P'Q')$, then quadrilateral $ABQ'P'$ is cyclic.

(PQ) is parallel to $(P'Q')$ if and only if the points A, P', Q' and B are cocyclic.

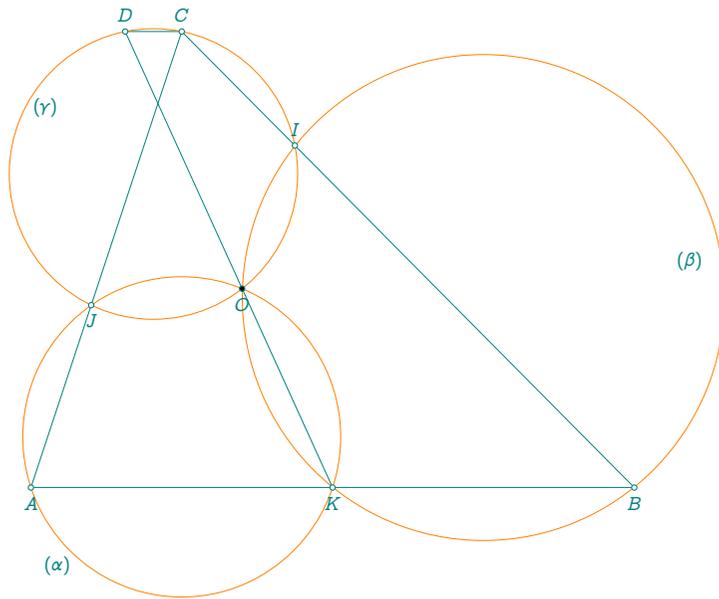It is sufficient to show that $\widehat{ABQ'} = \widehat{AP'Q'}$.

Figure 74: Reim 2b

K, O the intersection points of (α) and (β),
I one of the two points of intersection of (β) and (γ),
J one of the two points of intersection of (γ) and (α),
A a point of $(alpha)$,
B the second point of intersection of (AK) with (β)
C the second point of intersection of (BI) with (γ).
Let D be the second point of intersection of the line (KO) with (γ) Reim's theorem applied to the circles (α) and (γ).
The circles (γ) and (β)
Given a cyclic quadrilateral $ABFE$ and points G and H on the extensions of (EA) and (FB), respectively. If $GH \parallel EF$, then quadrilateral $ABHG$ is cyclic.
Reim's theorem $(CD) \parallel (AB)$

7.21 Three chords

β, γ two intersecting circles A, B the points of intersection of β and γ, C, D two points of γ, E, F two points of β, and I the point of intersection of (AB) and

(CD).

Question : C, D, E, F are cocyclic if, and only if, (EF) passes through I. Let us note: (α) the circle passing through C, D, E, F. (δ) the circle passing through B, F, D. and A', C', E' the second points of intersection of $(AB), (CD), (EF)$ with (δ).

Reim's theorem $(AC) \parallel (A'C')$ $(CE) \parallel (C'E')$ $(EA) \parallel (E'A')$

According to Desargues "The weak theorem" (Cf. Appendix) applied to homothetic triangles EAC and $E'A'C'$, (EE') passes through I. Conclusion: (EF) passes through I.

Monge's statement: when three circles intersect in pairs, the three intersecting lines are concurrent. Statement: If two circles are secant and the common line passes through the point of intersection of a secant of one and a secant of the other, then the four points of intersection are cocyclic.

Figure 75: three chords version 1

Figure 76: three chords version 2

7.22 South Pole

> **TH du Pole Sud**
>
> *The (interior) bisector from the vertex of a triangle and the perpendicular bisector of the opposite side intersect on the circumscribed circle.*

Proof: The triangle $\widehat{BOC}$ is isosceles because $\widehat{CBO}$ and $\widehat{BCO}$ are equal ($\widehat{CBO} = \widehat{OAB} = \widehat{BCO}$). O is thus on the bisector of $[BC]$ and the arcs $\overset{\frown}{BO}$ and $\overset{\frown}{OC}$ have the same length. O is called the "south pole" of triangle ABC.

Then let us show that the triangle BIO is isosceles at O. Let $\widehat{BAC} = \beta$, $\widehat{ABC} = \gamma$ and $\widehat{BCA} = \alpha$, then $\widehat{BIO} = 180° - \alpha - \dfrac{\beta}{2} - \dfrac{\gamma}{2} = \dfrac{\beta}{2} + \dfrac{\gamma}{2} = \widehat{IBO}$ so triangle BIO is isosceles at L. We deduce that $OB = OI = OC$ and B, I and C are on the circle of center O.
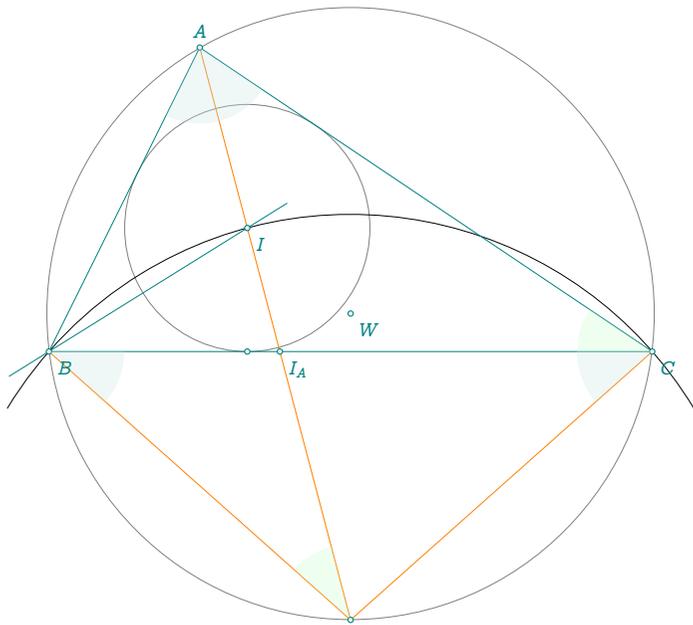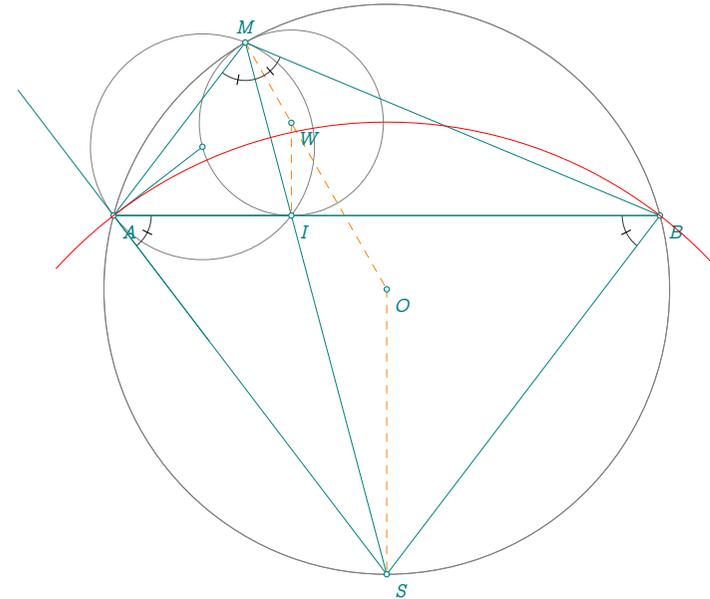

Figure 77: South Pole

7.22.1 Circle in circular segment

Il existe un cercle tangent $\mathcal{C}$ au cercle principal en M et I. Les points M, I et S (pôle sud) sont alignés. Le cercle $\mathcal{C}(S,A)$ est orthogonal au $\mathcal{C}$. Les tangents en leur points de contact passent par S.

Theorem "South Pole" : The bisector at M passes through the south pole S.
MWI = MOS so IMW = SMO and M, I and S are aligned.
The inversion of center S and circle $\mathcal{C}_{S,A}$ transforms the circle $\mathcal{C}_{O,M}$ into a straight line (A,B). The image of M is I, which is probably sufficient to justify the alignment. The circle $\mathcal{C}_{W,M}$ is globally invariant and therefore orthogonal to $\mathcal{C}_{S,A}$. The tangents at the points of intersection pass through S.

7.23 Tangent-Chord Theorem

> **Tangent-Chord Theorem**
>
> *The Tangent-Chord Theorem states that the angle formed between a chord and a tangent line to a circle is equal to the inscribed angle on the other side of the chord.*
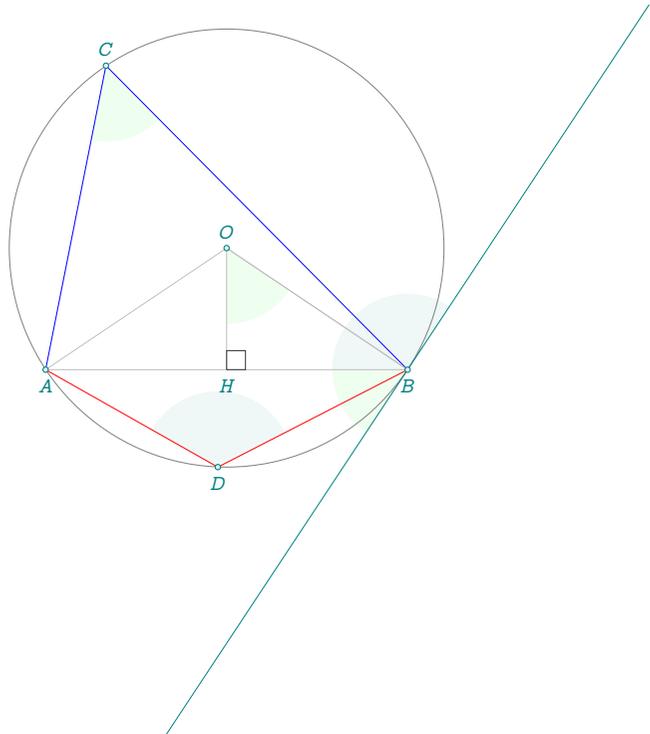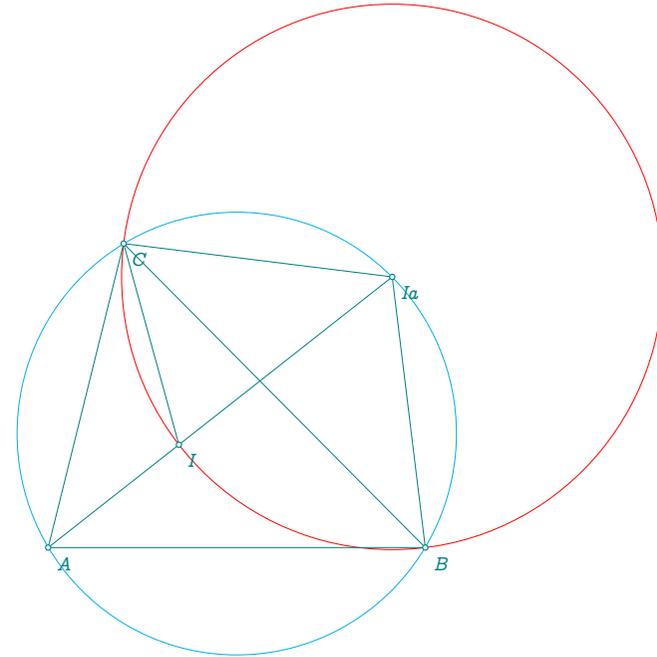
7.24 Circle through incircle

Figure 78: Tangent-Chord Theorem

$\widehat{IaB} = \widehat{IaC}$ so (AIa) is a bisector of $\widehat{BAC}$

Then $\widehat{ICB} = 1/2\widehat{IIaB} = 1/2\widehat{AIaB} = 1/2\widehat{ACB}$

So (CI) is a bisector of $\widehat{ACB}$

Conclusion: I is the incenter of ABC