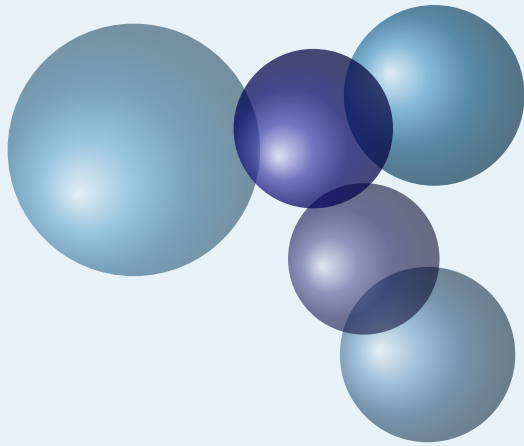


Sangakus with Lua 1.00c



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<http://altermundus.fr>

Sangakus with Lua

Alain Matthes

This document is a little gallery of some simple sangakus. I made all these pictures with my packages **tkz-elements V5.10c** and **tkz-euclide V5.13c** (option: mini). Sangaku (or san gaku) are colorful wooden tablets, often hung in Shinto shrines and sometimes in Buddhist temples in Japan, presenting typical and elegant mathematical problems. These problems belong to Japanese mathematics (wasan). The earliest sangaku found date back to the beginning of the 17th century.

☞ I would like to thank Till Tantau for the beautiful L^AT_EX package TikZ.

☞ You will find most of the examples in this document and many other examples on my website: altermundus.fr

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Contents

List of Figures

1 Sangaku - References

- “Japanese Temple Geometry”, a Scientific American article by Tony Rothman written in co-operation with Hidetoshi Fukagawa (Sci. Amer. 278, 85–91, May 1998);
- Japanese Temple Geometry Problems, the book by H. Fukagawa and D. Pedoe (1989);
- <http://www.sangaku.info/>;
- <http://mathworld.wolfram.com>;
- <http://www.wasan.jp/english/>;
- <http://www.cut-the-knot.org/pythagoras/Sangaku.shtml>.

2 Sangaku in a square - A very simple sangaku.

2.1 The picture

Find a relationship between the radius of the orange circle and the side of the square.

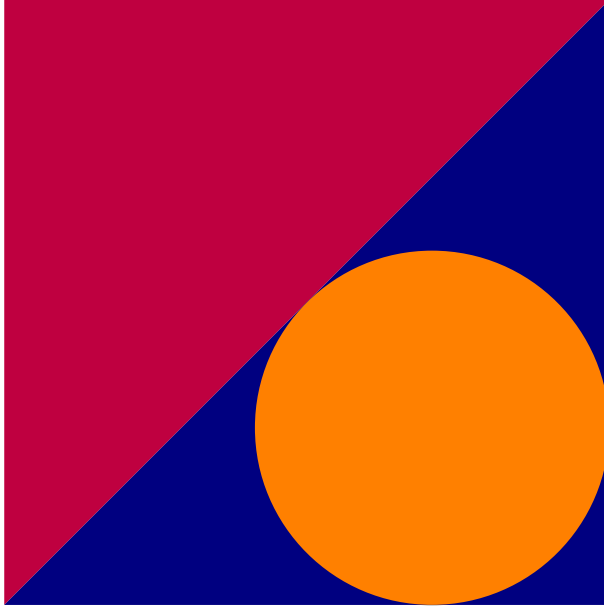


Figure 1: Sangaku 3b in a square

2.2 Explanation

Firstly, we find the relationship between the inradius and the sides of a right triangle.

If r is the inradius of a right triangle with legs a and b and hypotenuse c , then

$$r = \frac{1}{2}(a + b - c).$$

Let ABC be a right triangle, with the right angle at C , as shown in the figure. Let a, b and c be the lengths of the three sides; c is the length of the hypotenuse.

Let r and p be the radius of the incircle and the perimeter of the triangle. a, b and c can be regarded in relation to r and they may be expressed with r : $a = r + (a - r)$, $b = r + (b - r)$ and $c = (a - r) + (b - r)$. In a right triangle, we have the relation $r = p/2 - c$. From the diagram, the hypotenuse AB is split in two pieces: $(a - r)$ and $(b - r)$; the length of the hypotenuse is $c = (a - r) + (b - r)$.

The perimeter is a function of r :

$$p = a + b + c = r + (a - r) + r + (b - r) + (a - r) + (b - r) = 2a + 2b - 2r$$

so we can express r with p and c :

$$2r = a + b - c = p - 2c \text{ and } r = \frac{p}{2} - c = \frac{a + b - c}{2}$$

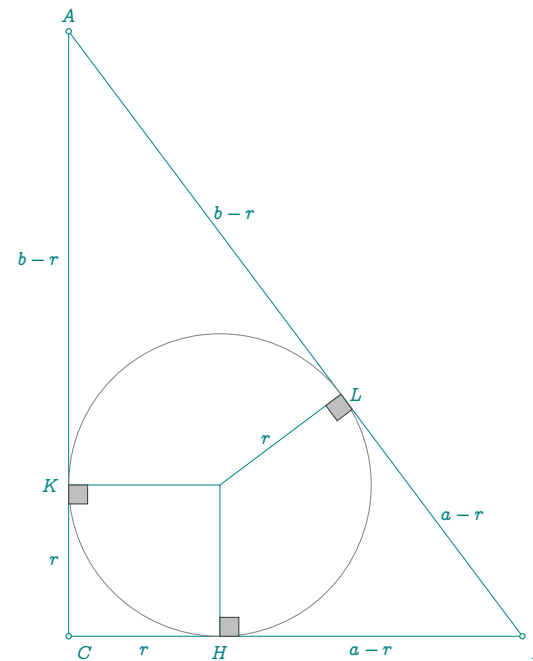


Figure 2: Sangaku 3a

Now, let ABC be an isosceles right triangle with $CA = CB = a$; then $AB = \sqrt{2}a$ and $a + b - c = 2a - \sqrt{2}a$.

So the inradius in this case is

$$r = \frac{2 - \sqrt{2}}{2}a$$

Now we can obtain the incenter without the bisectors. In the square $ABCD$ with center I , place H on $[CB]$ with $CH = CI$; then $HB = a - \frac{\sqrt{2}}{2}a = r$. The circle with center H passing through B meets (BI) at M , the incenter of ABC (we have $BM = \sqrt{2}r$).

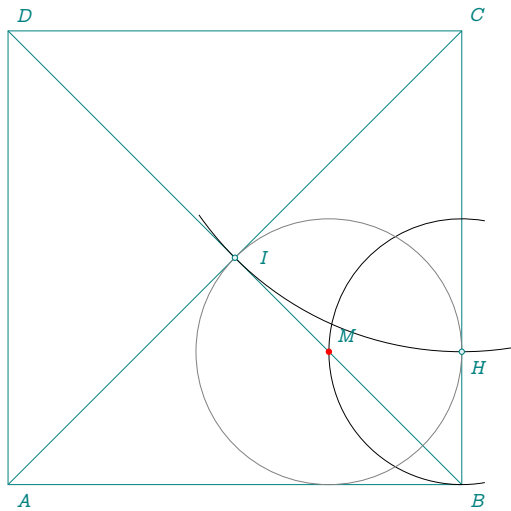


Figure 3: Sangaku 3c

3 Sangaku in a square - Circle and semicircle

Find a relationship between the radius of the orange circle and the side of the square.

3.1 The picture

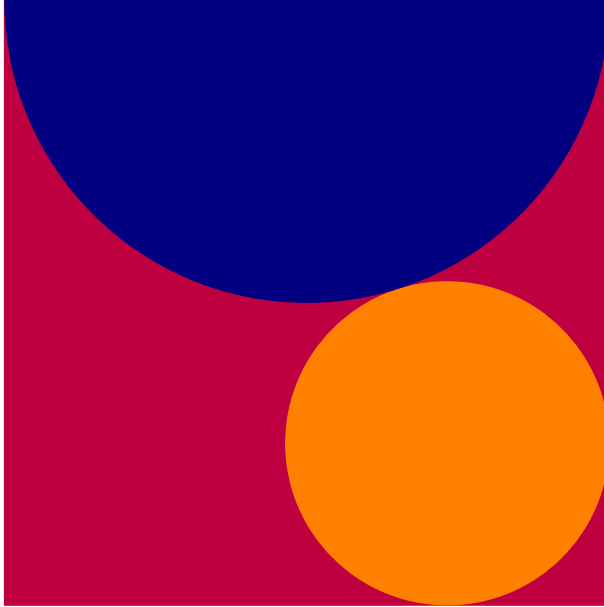


Figure 4: Sangaku 5a in a square

3.2 Explanation

FNE is a right triangle with hypotenuse $[EF]$. We have $EN^2 + NF^2 = EF^2$ by the Pythagorean theorem. In terms of a and r , the theorem appears as

$$\left(\frac{a}{2} - r\right)^2 + (a - r)^2 = \left(\frac{a}{2} + r\right)^2$$

which is equivalent to

$$r^2 - 4ar + a^2 = r^2 - 4ar + 4a^2 - 3a^2 = (r - 2a - \sqrt{3}a)(r - 2a + \sqrt{3}a) = 0$$

And finally

$$r = a(2 - \sqrt{3}) < a$$

3.3 Construction

It is easy to prove that $BJ = a \left(1 - \frac{\sqrt{3}}{2}\right)$ and $BL = a(2 - \sqrt{3}) = 2BJ$.

First we need to find the point I (third vertex of the equilateral triangle built on $[CD]$), then J . K is the intersection of (IJ) and the bisector of the angle $\widehat{ABC}$. Finally, F is the symmetric point of B with respect to K .

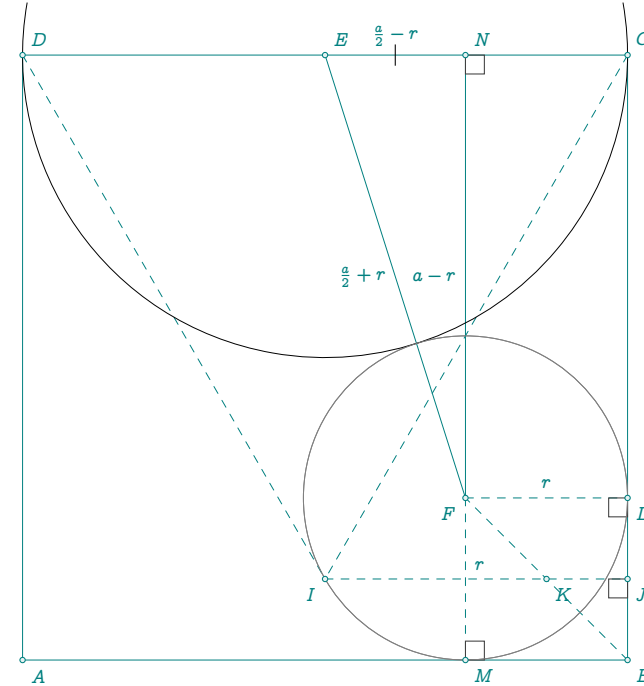


Figure 5: Sangaku 5b

4 Sangaku in a square - two inscribed circles

In the following diagram, a triangle is formed by a diagonal of a square and a line that joins a vertex of the base with the midpoint of the opposite side. Find the radii of the two inscribed circles.

4.1 The picture

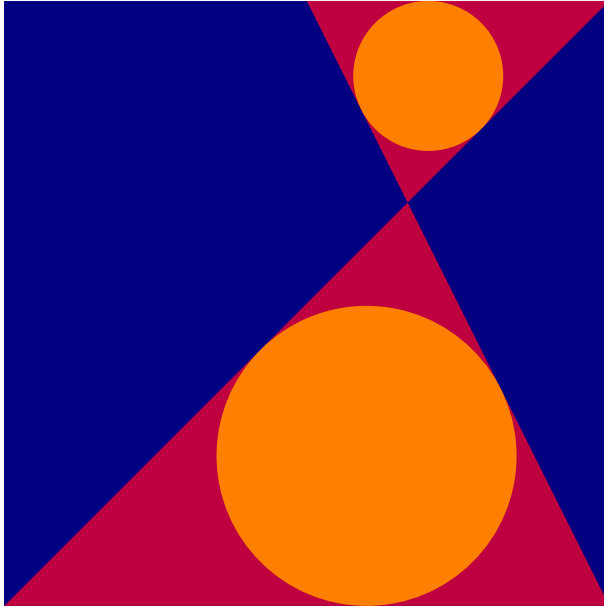


Figure 6: Sangaku 4 in a square - two inscribed circles

4.2 Explanation and construction

EC is parallel to the base AB and is half as long, which implies that the two triangles IEC and IBA are similar (ratio $1/2$). I divides the segments $[PQ]$, $[AC]$ and $[BE]$ in the ratio $2 : 1$, so that

$$IP = \frac{2}{3}QP = \frac{2a}{3}$$

$$AI = \frac{2}{3}AC$$

Thus assuming $AB = a$, we have $AC = \sqrt{2}a$, $AI = \frac{2\sqrt{2}}{3}a$ and $BI = \frac{2}{3}BE$. We can apply the Pythagorean theorem to find BE :

$$BE = \frac{\sqrt{5}}{2}a \quad \text{this means that} \quad BI = \frac{\sqrt{5}}{3}a$$

In any triangle, $r \times p = s \times h$, where r is the inradius, p the perimeter, s a side and h the corresponding altitude.

For the triangle IAB , with $s = AB = a$ and $h = IP$:

$$r \left(1 + \frac{\sqrt{5}}{3} + \frac{2\sqrt{2}}{3} \right) a = a \times \frac{2a}{3}$$

from which r is found:

$$r = \frac{2a}{3 + 2\sqrt{2} + \sqrt{5}}$$

The triangle IEC is the image of IBA in a ratio $1/2$, so the radius of the small circle is

$$r' = \frac{r}{2} = \frac{a}{3 + 2\sqrt{2} + \sqrt{5}}$$



5 Sangaku in a square - Two equilateral triangles

Here is an elegant sangaku that requires both geometric and algebraic skills and some perseverance:

Two equilateral triangles are inscribed into a square as shown in the diagram. Their side lines cut the square into a quadrilateral and a few triangles. Find a relationship between the radii of the two incircles shown in the diagram.

5.1 The figure

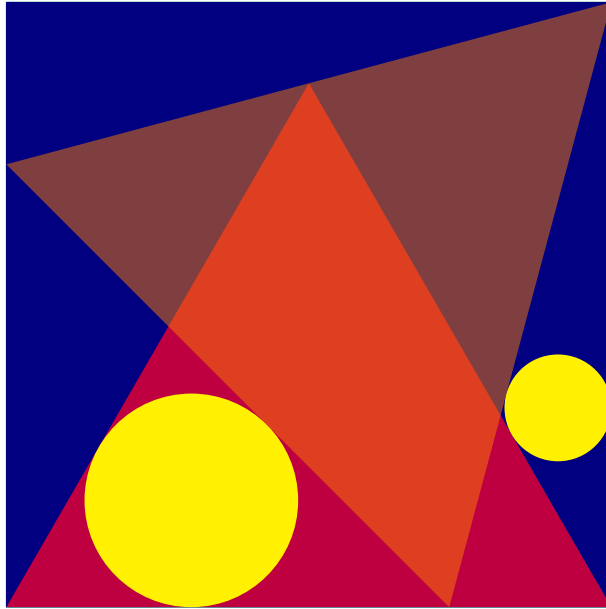


Figure 8: Sangaku 2a in a square - Two equilateral triangles

Firstly, we need to prove that two equilateral triangles can be inscribed in a square as shown in the diagram. A general theorem exists, but it is nice to find a solution in this particular case. Let $ABCD$ be a square and BCM an equilateral triangle. The line (DM) intersects $[AB]$ at point N . Then we construct a point L on the side $[BC]$ such that $\widehat{NDL} = 60^\circ$. The triangle MCD is isosceles, with the two sides MC and CD of the same length a . It follows that

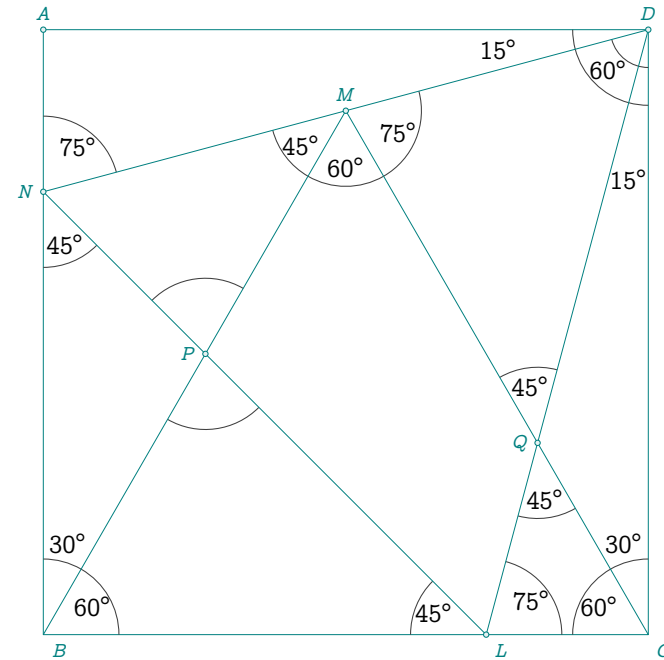


Figure 9: Sangaku 2b

$$\widehat{MDC} = \widehat{DMC} = 75^\circ \text{ because } \widehat{MCD} = 30^\circ$$

Now we can determine all the angles:

$$\widehat{LDC} = 15^\circ \text{ so } \widehat{ADN} = 15^\circ \text{ and } \widehat{AND} = 75^\circ$$

$$AN = LC, \text{ then } BL = BN \text{ and } \widehat{BLN} = \widehat{LNB} = \widehat{NMB} = \widehat{MQD} = \widehat{LQC} = 45^\circ$$

5.2 Explanation

We can see that the angles $\widehat{DNL}$ and $\widehat{NLD}$ both measure 60° . DNL is an equilateral triangle and it is the largest equilateral triangle which can be inscribed in the square (Madachy 1979). We prove later that its side is $s = (\sqrt{6} - \sqrt{2})a$.

We need some preliminaries to find the ratio between the radii of the two incircles shown in the first diagram.

thus

$$BP = \frac{2h}{\sqrt{3}}$$

HPL is an isosceles right triangle. The hypotenuse PL has length $\sqrt{2}h$. Finally, the relation between BL and h is obtained like this:

$$BL = BH + HL = \frac{h}{\sqrt{3}} + h$$

On the other hand

$$BL = BN = (\sqrt{3} - 1)a$$

The inradius r_1 of BLP is

$$r_1 = \frac{sh}{p} = \frac{(\sqrt{3} - 1)ah}{\frac{2h}{\sqrt{3}} + \sqrt{2}h + \frac{h}{\sqrt{3}} + h} = \frac{(\sqrt{3} - 1)a}{1 + \sqrt{2} + \sqrt{3}}$$

For CQD , similarly, the inradius r_2 can be found with

$$r_2 = \frac{ah}{p}$$

with $p = CQ + QD + DC$ and $h = QG$.

The triangles DQG and DLC are similar, so

$$\frac{QD}{DL} = \frac{QG}{LC} = \frac{h}{(2 - \sqrt{3})a}$$

from which

$$QD = \frac{(\sqrt{6} - \sqrt{2})ah}{(2 - \sqrt{3})a} = (\sqrt{6} + \sqrt{2})h$$

We continue: $\widehat{QCG} = 30^\circ$, so

$$\frac{QG}{QC} = \frac{h}{QC} = \sin(30^\circ) = \frac{1}{2}$$

from which

$$QC = 2h$$

and finally $DC = DG + GC$ with

$$\frac{h}{DG} = \tan(15^\circ) = 2 - \sqrt{3}$$

So

$$DG = \frac{h}{2 - \sqrt{3}} = (2 + \sqrt{3})h$$

and

$$\frac{GC}{QC} = \frac{GC}{2h} = \frac{\sqrt{3}}{2}$$

Thus we can find

$$GC = \sqrt{3}h$$

Finally

$$p = (\sqrt{6} + \sqrt{2})h + 2h + (2 + \sqrt{3})h + \sqrt{3}h$$

The second radius is

$$r_2 = \frac{ah}{p} = \frac{ah}{(\sqrt{6} + \sqrt{2})h + 2h + (2 + \sqrt{3})h + \sqrt{3}h} = \frac{a}{4 + \sqrt{2} + 2\sqrt{3} + \sqrt{6}}$$

Since $4 + \sqrt{2} + 2\sqrt{3} + \sqrt{6} = (1 + \sqrt{2} + \sqrt{3})(1 + \sqrt{3})$ and $(\sqrt{3} - 1)(\sqrt{3} + 1) = 2$, we conclude that $r_1 = 2r_2$.

5.3 Construction

There is no problem. First we draw the square $ABCD$ and the equilateral triangle BCM , then we draw the line (DM) . N is the intersection of (DM) with (AB) . We draw the equilateral triangle DNL . P and Q are intersections of sides of the equilateral triangles. Finally we draw the incircles of the triangles BPL and CQD .

6 Sangaku in a square - 3 Arcs and 1 Circle

This sangaku, which asks for the relative radii of the circles shown, can be solved by an application of the Pythagorean theorem. Find a relationship between the radius of the yellow circle and the side of the square.

6.1 The picture

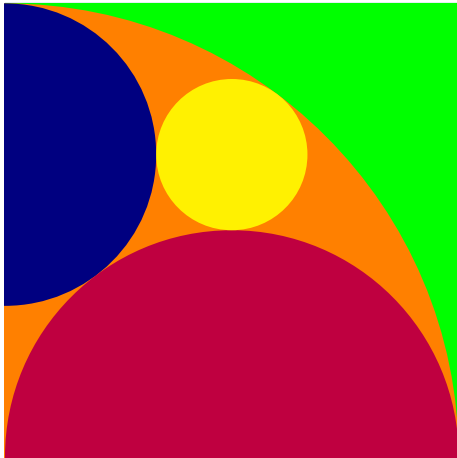


Figure 11: Sangaku 6a in a square

6.2 Explanation and construction

Assume the radius AM equals r and the side of the square is a .

step 1. Firstly, $ME^2 = BE^2 + BM^2$ and the two circles are tangent if $ME = r + \frac{a}{2}$. The equality becomes

$$\left(r + \frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2 + (a - r)^2$$

$$r = \frac{a}{3}$$

we have

$$ME = BK = \frac{5}{6}a$$

step 2.

$$KQ = KE - QE = \frac{2a}{3} - \frac{a}{2} = \frac{a}{6}$$

and

$$MK - \frac{a}{3} = \frac{a}{2} - \frac{a}{3} = \frac{a}{6}$$

The circle K is tangent to the circle E and to the circle M .

step 3. The little circle with center K is tangent to the circle B .

$$a - \frac{5a}{6} = \frac{a}{6}$$

step 4. Finally, a good method to obtain K is to place I on $[DC]$ such that $DI = \frac{a}{4}$. Then (BI) is tangent at G to the circle with diameter $[AD]$ (center F), with $GI = DI = \frac{a}{4}$, $FG = \frac{a}{2}$ and (FG) perpendicular to (BG) . (FG) intersects (DC) at J such that $DJ = \frac{2a}{3}$. K is the common point of (AJ) and (EF) .

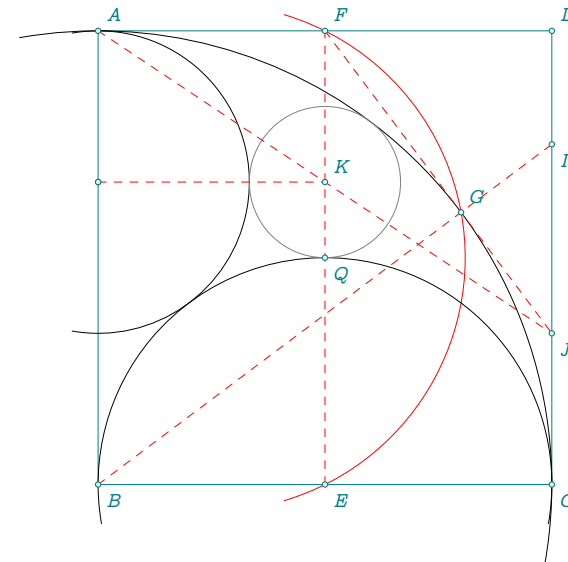


Figure 12: Sangaku 6b

7 Sangaku - Triangle and three circles

References

Weisstein, Eric W. "Circle Inscribing." From MathWorld—A Wolfram Web Resource

<http://mathworld.wolfram.com/CircleInscribing.html>

Alexander Bogomolny

<http://www.cut-the-knot.org/>

Construct the figure consisting of a circle centered at O , a second smaller circle centered at O_2 tangent to the first, and an isosceles triangle whose base $[AB]$ completes the diameter of the larger circle $[XB]$ through the smaller $[XA]$. Now inscribe a third circle with center O_3 inside the large circle, outside the small one, and on the side of a leg of the triangle. (from an 1803 sangaku found in Gumma Prefecture)

7.1 The figure

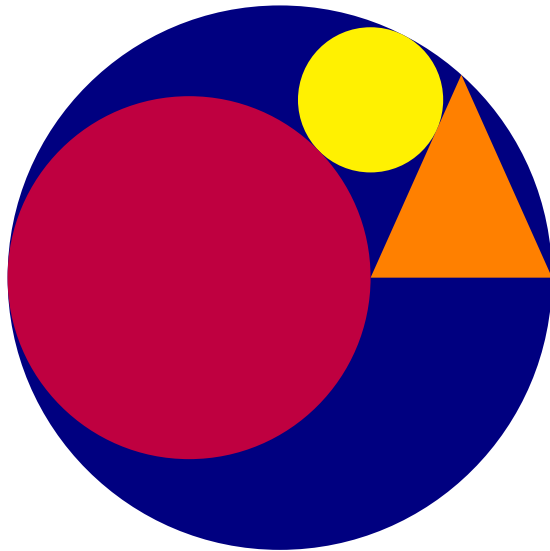


Figure 13: Sangaku - Circle Inscribing

7.2 Explanation

To find the explicit position and size of the circle, let the circle with center O have radius R , the circle with center O_2 radius r , and the circle with center O_3 radius a ; let $y = AO_3$.

$$(r + a)^2 = r^2 + y^2$$

$$(R - a)^2 = (R - 2r)^2 + y^2$$

Solving for a and y gives

$$a = 2r \frac{R - r}{R + r}$$

$$y = AO_3 = 2\sqrt{2Rr} \frac{\sqrt{R - r}}{R + r}$$

But now we need to prove that the circle is tangent to the line (AC) .

Let α be the angle $\widehat{ACD}$, which is also the angle $\widehat{O_3AC}$.

$$\sin(\alpha) = \frac{AD}{AC}$$

$OD = r$ and $AD = R - r$

OCD is a right triangle with hypotenuse $OC = R$. We have $OD^2 + CD^2 = OC^2$ by the Pythagorean theorem. In terms of r , the theorem appears as

$$r^2 + CD^2 = R^2$$

which is equivalent to

$$CD^2 = R^2 - r^2$$

and with the right triangle ADC and the Pythagorean theorem

$$AC^2 = AD^2 + CD^2 = (R - r)^2 + R^2 - r^2 = 2R(R - r)$$

finally

$$\sin(\alpha) = \frac{AD}{AC} = \frac{R-r}{\sqrt{2R(R-r)}} = \frac{\sqrt{R-r}}{\sqrt{2R}}$$

Let H be the projection of O_3 on the line (AC) , and d the length of O_3H :

$$\sin(\alpha) = \frac{O_3H}{AO_3} = \frac{d}{y} = \frac{d}{2\sqrt{2Rr} \frac{\sqrt{R-r}}{R+r}}$$

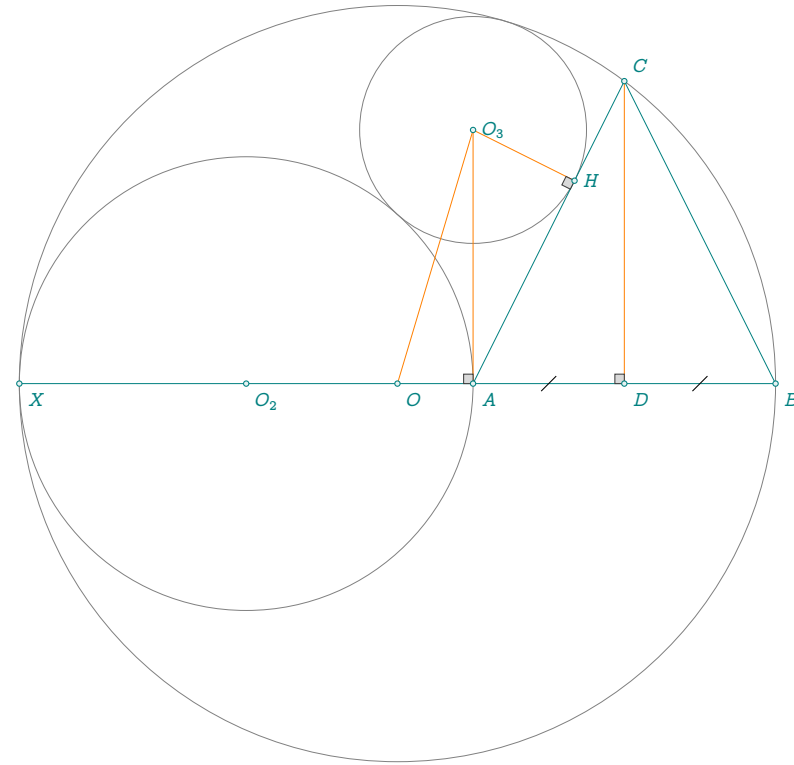
Using the two forms of $\sin(\alpha)$:

$$\frac{d}{2\sqrt{2Rr} \frac{\sqrt{R-r}}{R+r}} = \frac{\sqrt{R-r}}{\sqrt{2R}}$$

So

$$d = 2r \frac{R-r}{R+r} = a$$

so the circle with center O_3 is tangent to (AC) .



$$\begin{aligned}
OW^2 &= \left(1 - \frac{1}{au}\right)^2 \\
&= \left(u - \frac{1}{a}\right)^2 + \left(v - \frac{v}{au}\right)^2 \\
&= u^2 - \frac{2u}{a} + \frac{1}{a^2} + v^2 + \frac{v^2}{u^2 a^2} - \frac{2v^2}{ua} \\
OW^2 &= 1 - \frac{2u}{a} + \frac{1}{a^2} + \frac{1-u^2}{u^2 a^2} - \frac{2-2u^2}{ua} = \left(1 - \frac{1}{ua}\right)^2.
\end{aligned}$$

– S and V are the points of the line (QS) on the circle with center O , so their coordinates $(x; y)$ satisfy

$$ax + by = 1 \text{ and } \left(x - \frac{1}{a}\right)^2 + \left(y - \frac{v}{au}\right)^2 = \left(1 - \frac{1}{ua}\right)^2.$$

Using $a^2 + b^2 = 1$ and $u^2 + v^2 = 1$, we obtain

$$y^2 - 2\frac{av}{u}y + 2\frac{a}{u} - a^2 - 1 = 0$$

$$\text{The reduced discriminant is } \frac{a^2 v^2}{u^2} - 2\frac{a}{u} + a^2 + 1 = \frac{a^2}{u^2} - 2\frac{a}{u} + 1 = \left(1 - \frac{a}{u}\right)^2$$

This equation has two solutions:

$$y_1 = \frac{av}{u} - \left(1 - \frac{a}{u}\right) \text{ (negative, for } S) \text{ and } y_2 = \frac{av}{u} + \left(1 - \frac{a}{u}\right) \text{ (positive, for } V).$$

Finally

$$\begin{aligned}
QR &= -y_1 \\
&= 1 - \frac{a}{u} - \frac{av}{u}.
\end{aligned}$$

and

$$\begin{aligned}
PQ &= OP - OQ = OW - OQ = 1 - \frac{1}{ua} + \frac{v}{au} \\
PQ &= 1 - \frac{1}{au} + \frac{v}{au} = 1 - \frac{1-v}{au}.
\end{aligned}$$

So

$$(PQ - 1)(QR - 1) = \left(-\frac{a}{u} - \frac{av}{u}\right) \left(-\frac{1-v}{au}\right) = \frac{(1-v)(1+v)}{u^2} = 1.$$

Equivalently,

$$\frac{1}{PQ} + \frac{1}{QR} = 1 = \frac{1}{r}.$$

In the general case, some information is needed: the radius of the big circle and the positions of the points R and Q . In the code, R is given by its ordinate relative to O (`tucR`) and Q by the length QR (`tucQR`).

9 Sangaku - Harmonic mean

9.1 Harmonic mean of two numbers

$$\frac{2}{c} = \frac{1}{a} + \frac{1}{b}$$

which says that c is the harmonic mean of a and b .

For two numbers a and b , $\mathcal{A} = \frac{a+b}{2}$ and $\mathcal{G} = \sqrt{ab}$, and $\mathcal{H} = \frac{2ab}{a+b} = \frac{\mathcal{G}^2}{\mathcal{A}}$.

- Let a and b be two numbers such that $OA = a$ and $AB = b$. I is the center of the circle $\mathcal{C}$ with diameter $[OB]$. $[IK]$ is a radius perpendicular to (OB) . It is easy to prove that $IK = \mathcal{A}$.
- (AG) is the line perpendicular to (OB) at A , and G is a point of $\mathcal{C}$. OGB is a right triangle, so

$$AG^2 = OA \times AB$$

Finally we get $AG = \mathcal{G} = \sqrt{ab}$

GHA and GAI are right triangles with a common angle at G , so they are similar:

$$\frac{GH}{AG} = \frac{AG}{IG} \text{ and } \frac{GH}{\mathcal{G}} = \frac{\mathcal{G}}{IG}$$

In this case, we have (with $IG = \mathcal{A}$)

$$\mathcal{G}^2 = IG \times GH \text{ and } GH = \frac{\mathcal{G}^2}{\mathcal{A}}$$

Finally

$$GH = \mathcal{H}$$

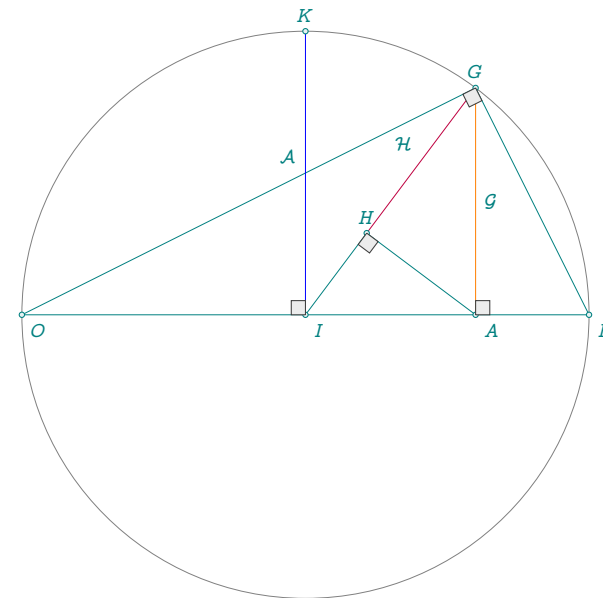


Figure 16: Sangaku 8a

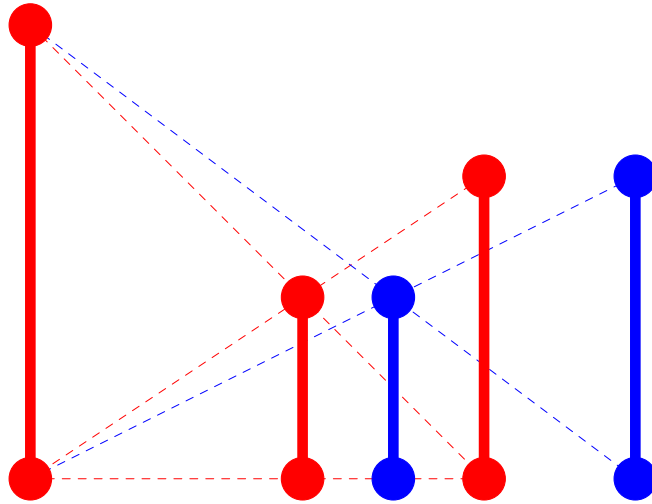


Figure 17: Sangaku 8b

9.2 Sangaku - Harmonic mean

Given two vertical segments $[AI]$ and $[BJ]$ ($AI = a$ and $BJ = b$), the intersection C of the diagonals is at a height that depends solely on AI and BJ . In fact

$$\frac{1}{CK} = \frac{1}{AI} + \frac{1}{BJ} = \frac{1}{a} + \frac{1}{b}$$

now if $B'J' = BJ$ then $C'K' = CK$.

Let's denote $AI = a$, $BJ = b$, $CK = c$, $IK = \alpha$ and $KJ = \beta$.

The triangles AIJ and CKJ are similar, as are the triangles BJI and CKI .

We have the proportions

$$\frac{c}{a} = \frac{\beta}{\alpha + \beta} \text{ and } \frac{c}{b} = \frac{\alpha}{\alpha + \beta}$$

We can add the two equalities

$$\frac{c}{a} + \frac{c}{b} = \frac{\beta}{\alpha + \beta} + \frac{\alpha}{\alpha + \beta} = 1$$

A division by c gives the desired result:

$$\frac{1}{c} = \frac{1}{a} + \frac{1}{b}$$

which says that c is half the harmonic mean of a and b .

10 Sangaku - Three Tangent Circles

10.1 Main method

<http://www.cut-the-knot.org/pythagoras/TangentCirclesSangaku.shtml>;

[Fukagawa & Pedoe 1.1.2]

Given three circles tangent to each other and to a straight line, express the radius of the middle circle via the radii of the other two. This problem was given as a Japanese temple problem on a tablet from 1824 in the Gunma Prefecture (MathWorld). This problem, too, requires nothing more than a few applications of the Pythagorean theorem. (Alexander Bogomolny)

10.1.1 The figure

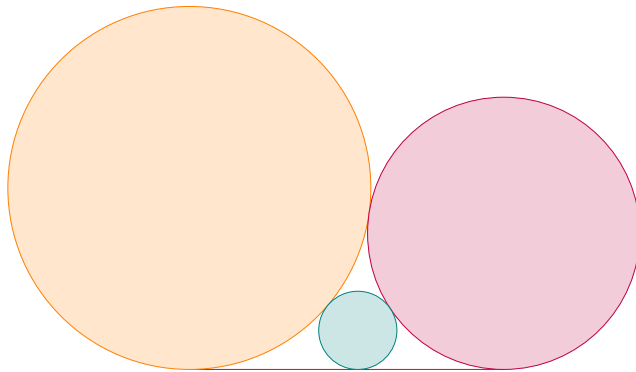


Figure 18: Sangaku 9a - Three Tangent Circles

10.1.2 Lemma

Given two circles tangent to each other and to a straight line at points E and F , we can express EF with the radii r_a and r_b of the two circles:

$$EF = 2\sqrt{r_a r_b}$$

Proof: ABH is a right triangle with hypotenuse $AB = r_a + r_b$. By the Pythagorean theorem

$$\begin{aligned} AB^2 &= AH^2 + HB^2 \\ &= AH^2 + d^2 \end{aligned}$$

If $r_a > r_b$, then $AH = r_a - r_b$ and

$$(r_a + r_b)^2 = (r_a - r_b)^2 + d^2$$

Finally $d^2 = 4r_a r_b$ and $d = 2\sqrt{r_a r_b}$.

Now we have two circles ($r_a > r_b$) tangent to each other and to a straight line, and we want to draw another circle tangent to both of them and to the same straight line:

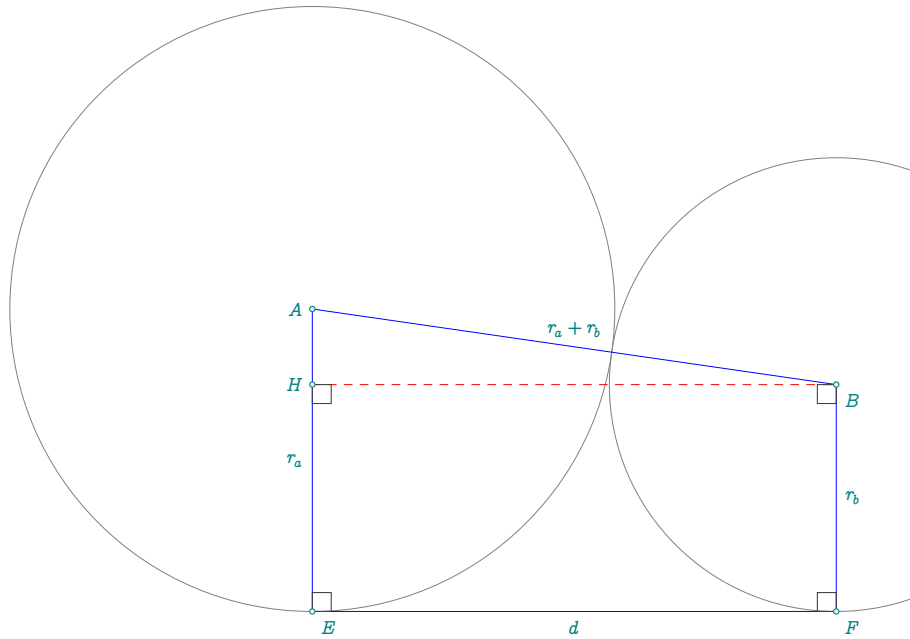
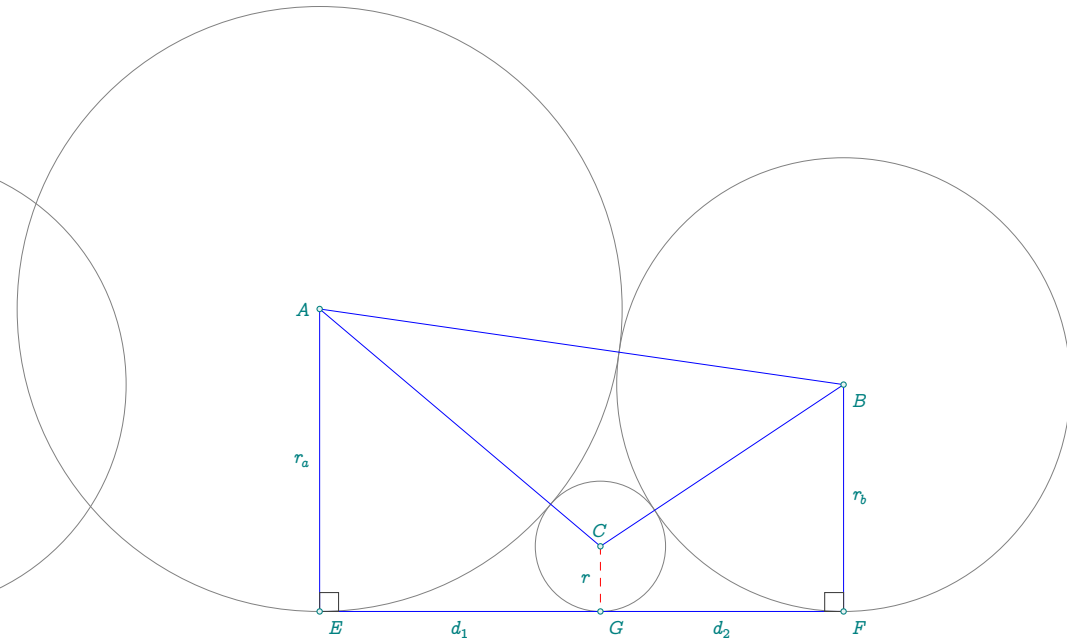


Figure 19: Lemma

Figure 20: Relation between r , r_a and r_b

We can apply the lemma to the circles $\mathcal{A} = (A, r_a)$ and $\mathcal{B} = (B, r_b)$, then to the circles $\mathcal{A}$ and $\mathcal{C} = (C, r)$, and finally to the circles $\mathcal{B}$ and $\mathcal{C}$.

We get four equations:

$$d = d_1 + d_2$$

$$d = 2\sqrt{r_a r_b}$$

$$d_1 = 2\sqrt{r_a r}$$

$$d_2 = 2\sqrt{r r_b}$$

Combining them, we get

$$2\sqrt{r_a r_b} = 2\sqrt{r_a r} + 2\sqrt{r r_b}$$

Now divide by $\sqrt{r}\sqrt{r_a}\sqrt{r_b}$ to obtain

$$\frac{1}{\sqrt{r}} = \frac{1}{\sqrt{r_a}} + \frac{1}{\sqrt{r_b}} \quad \text{so} \quad r = \frac{r_a r_b}{(\sqrt{r_a} + \sqrt{r_b})^2}$$

This is the condition between the radii.

10.1.3 How to make this construction with a ruler and a compass

Before starting the construction, let's find out what tools will be needed. We have established the following relationship between r , r_a and r_b :

$$r = \frac{r_a r_b}{(\sqrt{r_a} + \sqrt{r_b})^2}$$

which we can write:

$$r = \frac{r_a r_b}{(r_a + r_b + 2\sqrt{r_a r_b})}$$

So we need to represent $r_a r_b$ and $r_a + r_b + 2\sqrt{r_a r_b}$ before performing a division. This can be achieved in several ways. The only difficulty is to represent $2\sqrt{r_a r_b}$; we will see that it is the length of EF . Here's one:

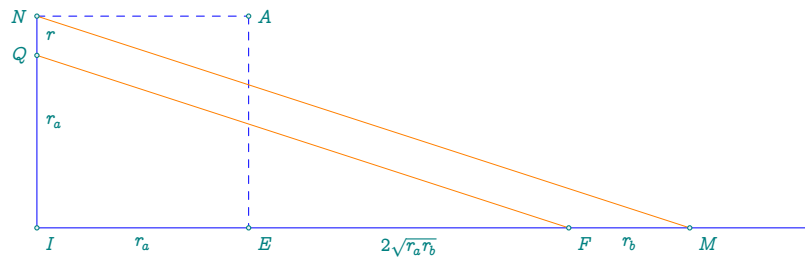


Figure 21: How to get r with ruler and compass

Let's see how to overcome the difficulty of plotting $2\sqrt{r_a r_b}$.

step 1. We draw a line (OX) and a circle with center A tangent to the line in E . The radius of this circle is $EA = r_a = 4$ cm.

step 2. Now we want to draw a second circle with center B and radius $r_b = FB = 2$ cm, tangent to the first circle and to the line (OX) at F .

We place a point P on the line (EA) such that $AP = r_a + r_b$ (so $EP = r_b$). The circle with diameter $[AP]$ intersects the (OX) axis at a point D such that $ED = \sqrt{r_a r_b}$.

AED and DEP are similar right triangles and we can write $\frac{ED}{EA} = \frac{EP}{ED}$, so $ED = \sqrt{r_a r_b}$. It is now easy to obtain F such that $EF = 2ED = 2\sqrt{r_a r_b}$. B is the symmetric point of P with respect to D . We can draw the circle with center B and radius BF .

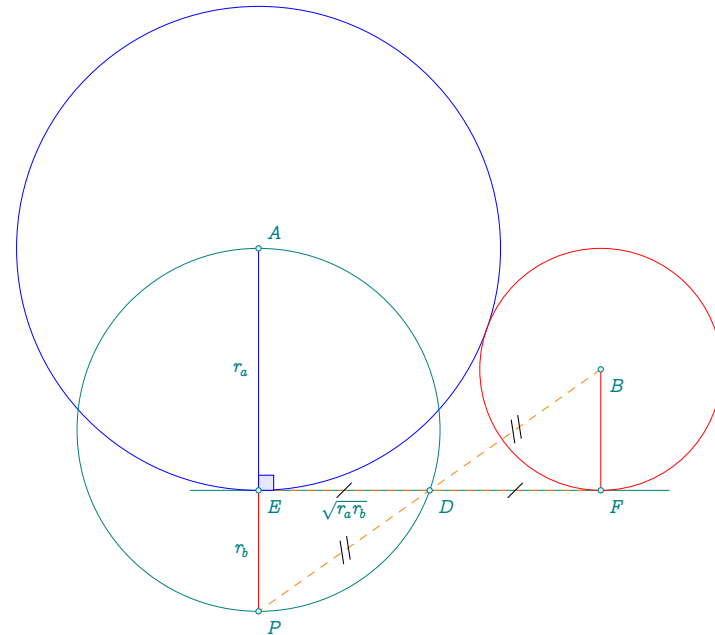


Figure 22: Sangaku 9b $EF = 2\sqrt{r_a r_b}$

step 3. Now we can use the sangaku about the harmonic mean.

The line (EB) intersects the line (AF) at a point K such that $(J$ is the projection of K on (EF))

$$\frac{1}{KJ} = \frac{1}{EA} + \frac{1}{FB} = \frac{1}{r_a} + \frac{1}{r_b}$$

$$\text{but } \frac{1}{\sqrt{r}} = \frac{1}{\sqrt{r_a}} + \frac{1}{\sqrt{r_b}}$$

$$\text{so } \frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{2}{\sqrt{r_a r_b}}$$

$$\frac{1}{r} = \frac{1}{KJ} + \frac{1}{\frac{\sqrt{r_a r_b}}{2}}$$

To find r , we need only to represent $\frac{\sqrt{r_a r_b}}{2}$.

$$ED = \sqrt{r_a r_b}, EF = 2\sqrt{r_a r_b} \text{ and } ED = EF/2, \text{ so } \frac{\sqrt{r_a r_b}}{2} = \frac{ED}{2}.$$

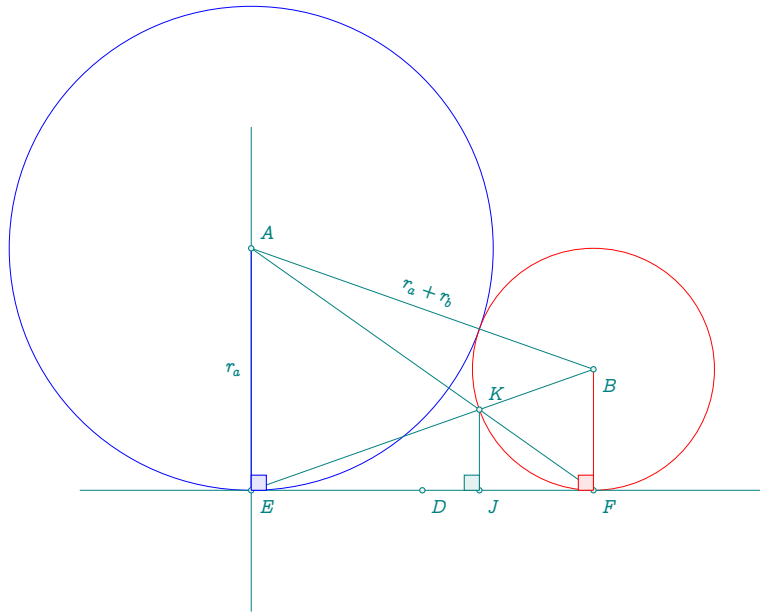


Figure 23: Sangaku 9c

step 4. $AB = r_a + r_b$ and the circles (A, r_a) and (B, r_b) are tangent. The lines (AF) and (EB) intersect at a point K . Let J be the orthogonal projection of K on (EF) . The previous sangaku about the harmonic mean allows us to write that

$$\begin{aligned} \frac{1}{KJ} &= \frac{1}{EA} + \frac{1}{FB} \\ &= \frac{1}{r_a} + \frac{1}{r_b} \end{aligned}$$

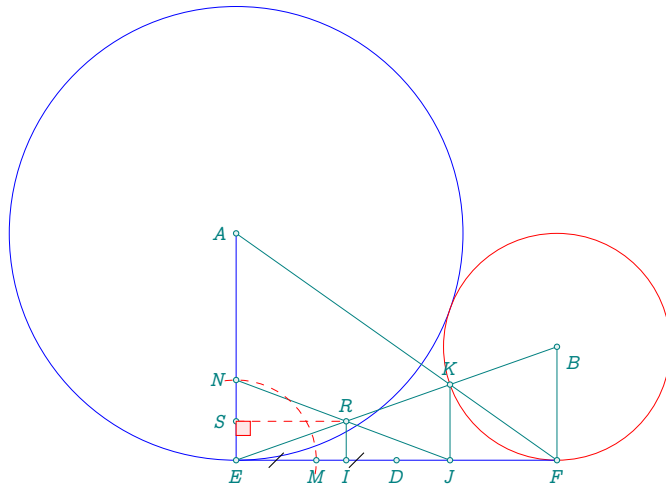
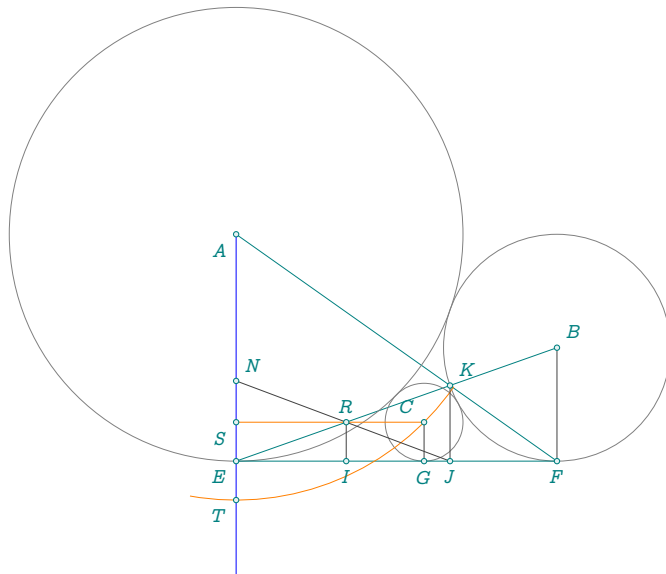
From $\frac{1}{\sqrt{r}} = \frac{1}{\sqrt{r_a}} + \frac{1}{\sqrt{r_b}}$ we get:

$$\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{2}{\sqrt{r_a r_b}} = \frac{1}{KJ} + \frac{1}{\frac{\sqrt{r_a r_b}}{2}}$$

In the next picture, $EM = EN = \frac{ED}{2} = \frac{\sqrt{r_a r_b}}{2}$ (N on (EA)). The line (EK) intersects the line (JN) at the point R . If I is the projection of R on (EF) , then $IR = r$: again, we use the sangaku about the harmonic mean.

$$\frac{1}{IR} = \frac{1}{KJ} + \frac{1}{EN}$$

step 5. Let S be the projection of R on (EA) , so $ES = IR = r$. The center C lies on the line (SR) and on the circle with center A and radius $r_a + r$. In the next picture, T is the symmetric point of S with respect to E , so $AT = r_a + r$, and C is the intersection of the circle (A, AT) with the line (SR) .

Figure 24: Sangaku 9d $ES = r$ Figure 25: Sangaku 9e First method to get C

10.2 Method Chen Bai

The next construction is based on a proposition of Chen Bai.

<https://math.stackexchange.com/questions/269951/sangaku-how-to-draw-those-three-circles-with-only-a-ruler-and-a-compass/>

10.2.1 The figure

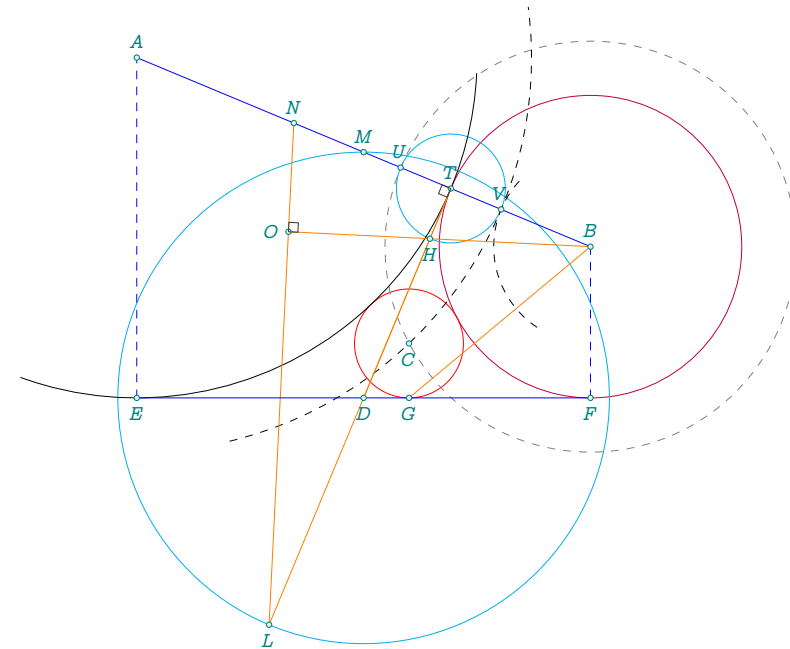


Figure 26: Sangaku 9f: Chen Bai

10.2.2 The explanation

T point of contact of A and B

M midpoint of $[AB]$, D midpoint of $[EF]$

N midpoint of $[AT]$

L intersection of (TD) with circle (D, M)

O projection of B on (NL)

H intersection of (TD) with (OB)

The triangles BTH and NLT are similar. We have

$$\frac{HT}{BT} = \frac{NT}{LT}$$

$$LT = LD + DT = DM + DT = \sqrt{r_a r_b} + \frac{r_a}{2} + \frac{r_b}{2} = \frac{(\sqrt{r_a} + \sqrt{r_b})^2}{2}$$

$$HT = \frac{BT \times NT}{LT} = \frac{r_b \times r_a / 2}{(\sqrt{r_a} + \sqrt{r_b})^2} = \frac{r_a r_b}{(\sqrt{r_a} + \sqrt{r_b})^2}$$

$$HT = r = \left(\frac{\sqrt{r_a} \sqrt{r_b}}{\sqrt{r_a} + \sqrt{r_b}} \right)^2$$

10.3 Construction with inversion

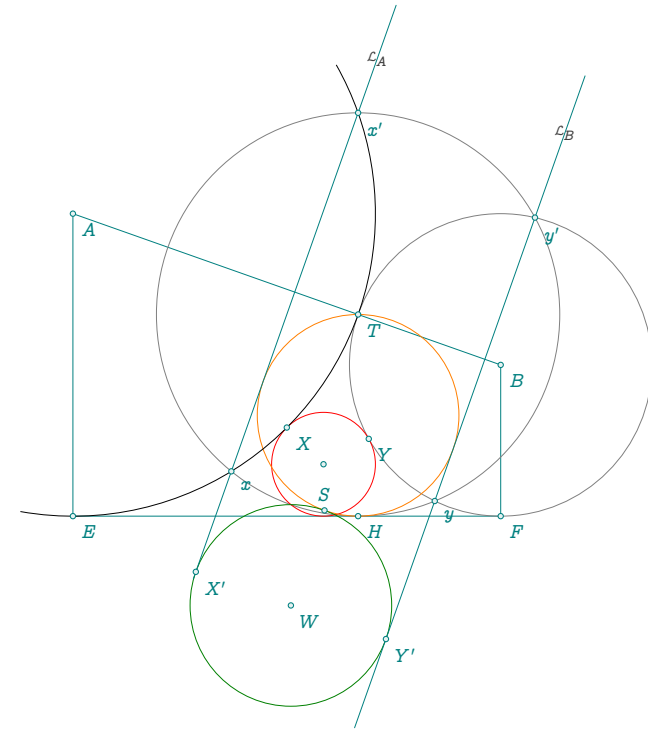


Figure 27: Sangaku 9g: With Inversion

Let us consider the circle $\mathcal{T}$ whose center T is the point common to the circles $\mathcal{A}$ and $\mathcal{B}$, and which is tangent to the line (EF) at H . It intersects the two circles at 4 points which define two lines $\mathcal{L}_A$ and $\mathcal{L}_B$ perpendicular to the line (AB) .

Consider the inversion with respect to the circle $\mathcal{T}$. The circles $\mathcal{A}$ and $\mathcal{B}$ have the lines $\mathcal{L}_A$ and $\mathcal{L}_B$ as inverses. The orange circle with diameter $[TH]$ is the inverse of the line (EF) . This circle is therefore tangent to the lines $\mathcal{L}_A$ and $\mathcal{L}_B$, since (EF) is tangent to the circles $\mathcal{A}$ and $\mathcal{B}$. Let the green circle $\mathcal{W}$ be tangent to the orange circle and to the lines $\mathcal{L}_A$ and $\mathcal{L}_B$, on the side of (EF) . Its inverse is a circle tangent to the circles $\mathcal{A}$ and $\mathcal{B}$ and to the line (EF) .

The last construction is the most complicated: we look for the intersection points of $\mathcal{W}$ and the circle $\mathcal{T}$. These points are invariant under the inversion

and belong to the sought circle. Then we draw the lines from T to X' and Y' , the points of contact of $\mathcal{W}$ with $\mathcal{L}_A$ and $\mathcal{L}_B$. These lines intersect the circles A and B at X and Y , the points of contact with the sought circle C .

10.4 Other method

The length of the segment $[RM]$ is $r_a + r_b + 2\sqrt{r_a r_b}$. Here is the figure:

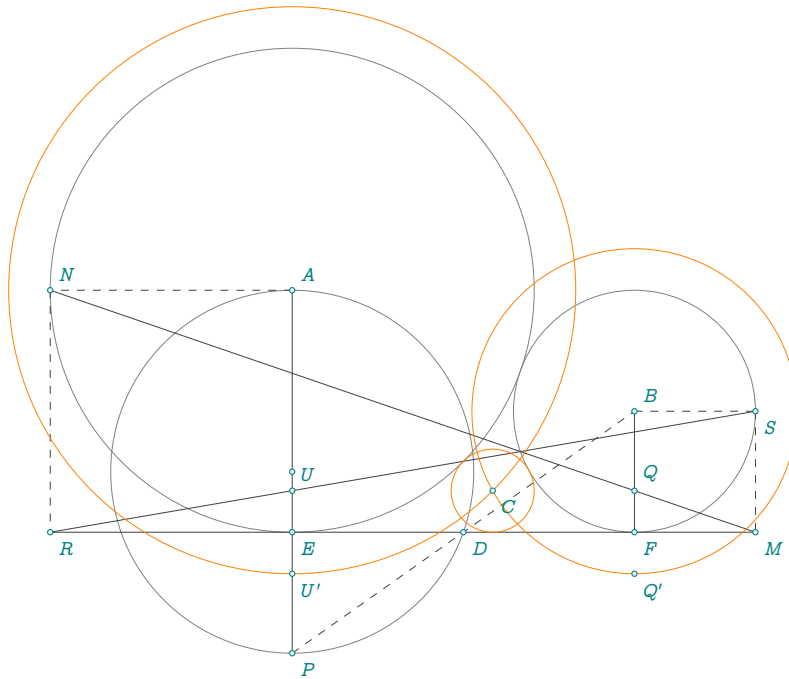


Figure 28: Sangaku 9h

10.5 Three tangent circles and a square

<http://www.gogeometry.com/school-college/3/p1228-three-tangent-circles-line-square.htm>

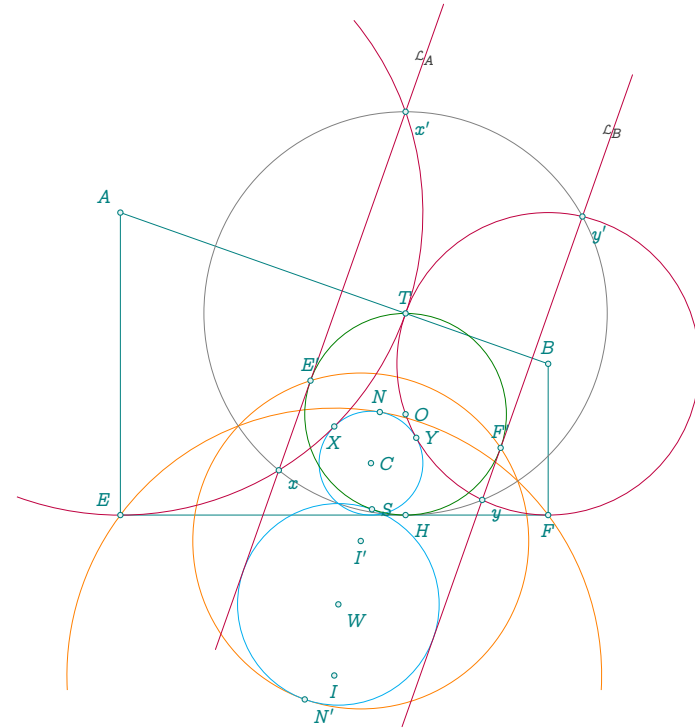


Figure 29: Sangaku 9i: Three circles and square

Circles A , B and C are tangent to each other and to a line at E , F and K respectively. The circle $\mathcal{I}$ passes through E and F and is tangent to the circle C at N . The tangent to the circle $\mathcal{I}$ at the midpoint M of the major arc EF meets (AE) and (BF) extended at H and G respectively. Prove that $EFGH$ is a square (Antonio Gutierrez, see figure ??).

- Step 1. Let's show how to obtain the circle with center I passing through E and F and tangent to the circle C (figure ??). E' and F' are the images of E and F by the previous inversion: they are the points of tangency

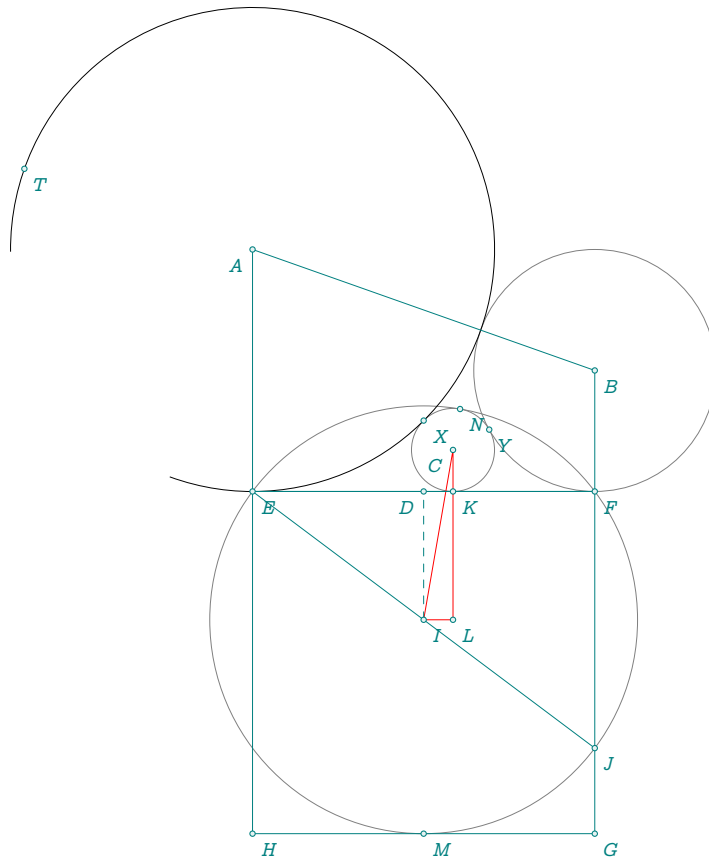


Figure 31: 3-4-5-triangle

(Solution : Sumith Peiris July 5, 2016)

Let $FJ = x$, $EJ = y$ and $EF = z$; the circles A , B , C have radii r_a , r_b , r and the radius of I is $R = y/2$ ($[EJ]$ is a diameter).

For the right triangle JFE : $x^2 + z^2 = y^2$

For the right triangle ILC with hypotenuse $IC = R - r$:

$$IC^2 = IL^2 + LC^2 = DK^2 + LC^2$$

$$(y/2 - r)^2 = (x/2 + r)^2 + (EK - z/2)^2$$

$$(y - 2r)^2 = (x + 2r)^2 + (2EK - z)^2$$

Using $y^2 = x^2 + z^2$:

$$(x + y)r = EK(z - EK) = 4r\sqrt{r_a r_b}$$

But $EK = 2\sqrt{r_a r}$, $KF = z - EK = 2\sqrt{r_b r}$ and $z = 2\sqrt{r_a r_b}$

Hence $x + y = 4\sqrt{r_a r_b} = 2z$

Hence $x + y = 2z$ and $x^2 + z^2 = y^2$.

We can eliminate z to find y as a function of x :

$$y^2 - x^2 = (x + y)^2/4 \text{ which yields } y - x = (x + y)/4$$

So $3y = 5x$, which implies $3z = 4x$.

Let $x = 3k$ where k is a real number; then

$x = 3k$, $y = 5k$ and $z = 4k$. The lengths of the sides are proportional to 3, 4 and 5.

Therefore JEF is a 3-4-5 triangle.

10.7 A Sangaku with an Egyptian Attachment

Alexander Bogomolny

www.cut-the-knot.org/Curriculum/Geometry/VariableTangentSangaku.shtml

Assume points E and F are fixed on a line and two circles are drawn touching (EF) at E and F and tangent to each other. A circle C is tangent to (EF) and externally to the two circles. Prove that, as the two circles change, the circle C remains tangent to a fixed circle through E and F . Moreover, the radius of the latter is $\frac{5}{8}EF$.

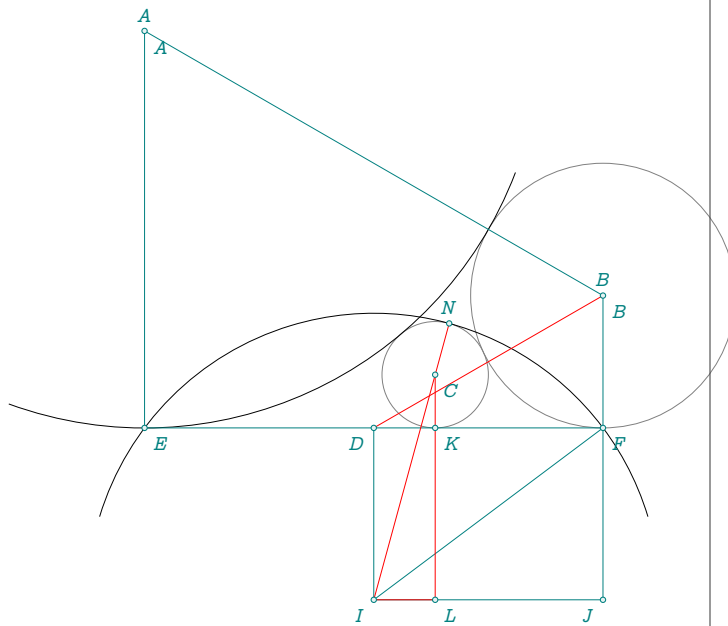


Figure 32: Egyptian triangle

In the previous section we saw that the triangle JEJ is an Egyptian (3-4-5) triangle. Here D is the midpoint of $[EF]$ and J is the intersection of the line (IL) with the line (BF) . The triangle DIF is similar to the triangle FJE (I is the midpoint of the hypotenuse $[EJ]$) and is therefore also an Egyptian triangle. Thus there exists a real t such that $DF = 4t$ and $IF = 5t$. As $DF = EF/2$, $t = EF/8$ and we easily deduce that $IF = \frac{5}{8}EF$. So the circle with center I passing through F is fixed.

10.8 Special position

If $r_a = 4$ and $r_b = 1$, then X' is on the line (EF) and X is the common point of $A, C, (TX')$ and the orange circle (O, H) . X is then the midpoint of $[TX']$.

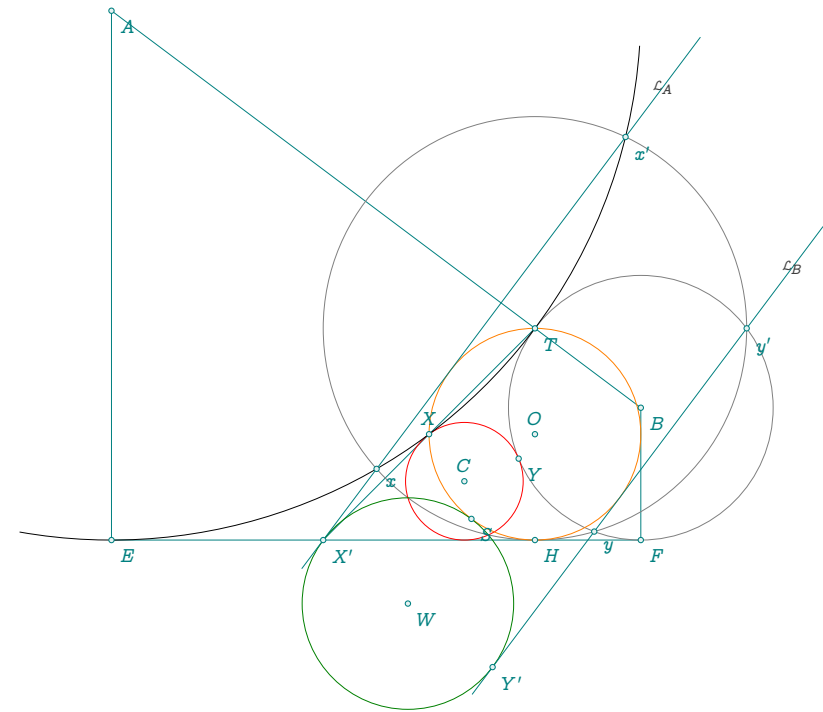


Figure 33: Sangaku 9m: Special position