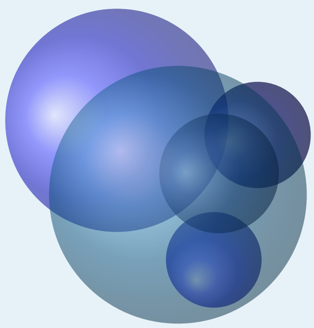
Straightedge-and-Compass

AlterMundus

Examples of Straightedge-and-Compass Constructions



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http://altermundus.fr

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This small treatise on straightedge-and-compass geometry presents a series of examples illustrating the use of the **tkz-elements** and **tkz-euclide** packages. Various themes are explored, including sacred geometry and constructible numbers. At present, these examples mainly serve as test cases to explore and validate the capabilities of the packages. Part of the text is adapted from the treatise `Pratique du Compas` by `Zacharie`, reproduced by Pierre Fournier.

– I would like to thank `Till Tantau` for the creation of Ti*k*Z, without which **tkz-euclide** would not exist;

– I would also like to thank `Yves Combes` for providing examples and for correcting several macros;

– My thanks also go to `Mark Wibrow` and `Christian Feuersänger` for their examples and for their help with **pgfkeys**;

– I would also like to thank `Jacques Yves André` for his assistance with fonts, and `Stéphane Pasquet` for his corrections;

– Among the various sources I have used, I should first mention *PRATIQUE DU COMPAS* by `Pierre Fournier`, available on the excellent Syracuse website: `http://melusine.eu.org/syracuse/metapost/compas.pdf`.

Please report typos or any other comments to this documentation to Alain Matthes

Contents

1 Notations

Here are the most common objects with their notation and how to construct them with **tkz-euclide**. The notations are the most commonly used in French literature.

1.1 Points

A point is represented geometrically by a small disc or a cross and is designated by a capital letter (except for construction points).

```
init_elements()
z.A = point(0,0)
z.B = point(5,1)
```


```
\directlua{dofile("tkz-geom-
lua/notation_1.lua")}

\begin{tikzpicture}
  \tkzGetNodes
  \tkzDrawPoint(A)
  \tkzDrawPoint[shape=cross,size=4](B)
  \tkzLabelPoints(A,B)
\end{tikzpicture}
```

1.2 Straight lines

1.2.1 Notation with two points

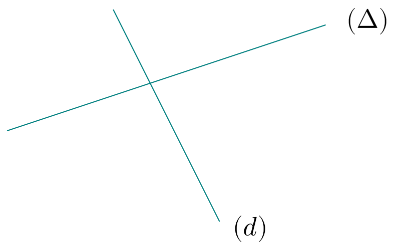
A straight line defined by two points A and B is denoted by (AB).

```
init_elements()
z.A = point(0,1)
z.B = point(3,2)
```


```
  \directlua{dofile("tkz-geom-
lua/notation_2.lua")}
\begin{tikzpicture}
  \tkzGetNodes
  \tkzDrawLine(A,B)
  \tkzLabelLine[pos=1.25,right](A,B){$(AB)$}
  \tkzDrawPoints(A,B)
  \tkzLabelPoints(A,B)
\end{tikzpicture}
```

1.2.2 Notation with a letter

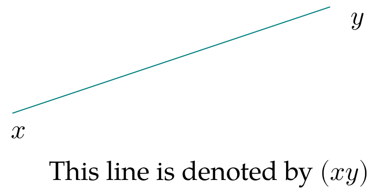
```
init_elements()
z.A = point(0,1)
z.B = point(3,2)
z.C = point(1,2)
z.D = point(2,0)
```




```
\directlua{dofile("tkz-geom-
lua/notation_3.lua")}

\begin{tikzpicture}
  \tkzGetNodes
  \tkzDrawLine(A,B)
  \tkzLabelLine[pos=1.25,right](A,B){$(\Delta)$}
  \tkzDrawLine(C,D)
  \tkzLabelLine[pos=1.25,right](C,D){$(d)$}
\end{tikzpicture}
```

1.2.3 Notation with two directions"

```
init_elements()
z.A = point(0,1)
z.B = point(3,2)
```



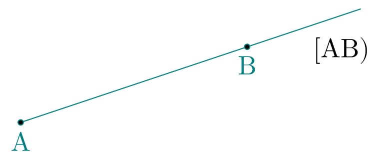

```
\directlua{dofile("tkz-geom-
lua/notation_2.lua")}
\begin{tikzpicture}
  \tkzGetNodes
  \tkzDrawLine(A,B)
  \tkzLabelLine[pos=1.25,below right](A,B){$y$}
  \tkzLabelLine[pos=-
0.25,below right](A,B){$x$}
  \tkzText(2,0){ This line is denoted by $(xy)$}
\end{tikzpicture}
```

1.3 Half-lines or rays

1.3.1 Notation with two points

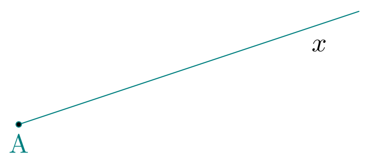
A ray defined by two points A and B is denoted by [AB).

```
init_elements()
z.A = point(0,1)
z.B = point(3,2)
```




```
\directlua{dofile("tkz-geom-
lua/notation_2.lua")}

\begin{tikzpicture}
  \tkzGetNodes
  \tkzDrawLine[add = 0 and  .5](A,B)
  \tkzLabelLine[pos=1.25,below right](A,B){$[AB)$}
  \tkzDrawPoints(A,B)
  \tkzLabelPoints(A,B)
\end{tikzpicture}
```

1.3.2 Notation with a dot and a direction



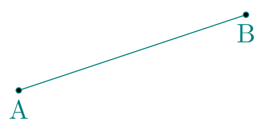

```
\directlua{
z.A = point(0,1)
z.B = point(3,2)
}
\begin{tikzpicture}
  \tkzGetNodes
  \tkzDrawLine[add = 0 and  .5](A,B)
  \tkzLabelLine[pos=1.25,below right](A,B){$x$}
  \tkzDrawPoints(A) \tkzLabelPoints(A)
\end{tikzpicture}
```

1.4 The segments

A segment is defined by two points and is denoted by $[AB]$

1.4.1 Example of segments

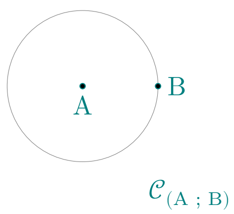
```
init_elements()
z.A = point(0,1)
z.B = point(3,2)
```




```
\directlua{dofile("tkz-geom-
lua/notation_2.lua")}
\begin{tikzpicture}
  \tkzGetNodes
  \tkzDrawSegment(A,B)
  \tkzDrawPoints(A,B)
  \tkzLabelPoints(A,B)
\end{tikzpicture}
```

1.5 The circles

1.5.1 Circle with center and through a point

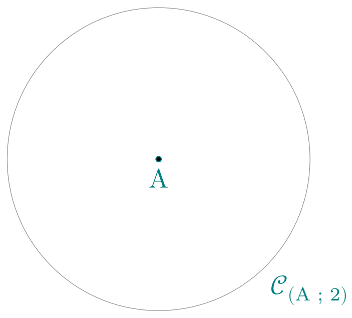



```
\directlua{
z.A = point(1,1)
z.B = point(2,1)
}
\begin{tikzpicture}
  \tkzGetNodes
  \tkzDrawCircle(A,B)
  \tkzDrawPoints(A,B)
  \tkzLabelPoints(A)
  \tkzLabelPoints[right](B)
  \tkzLabelCircle[below right=10pt](A,B)(-60)%
      {$\mathcal{C}_{(A~;~B)}$}
\end{tikzpicture}
```

1.5.2 Circle of given radius

```
init_elements()
z.A = point(0,1)
```

```
C.Aa = circle(through(z.A,2))
z.a = C.Aa.through
```




```
\directlua{dofile("tkz-geom-
lua/notation_5.lua")}
%or z.a = z.A + point(2, 0)
\begin{tikzpicture}
  \tkzGetNodes
  \tkzDrawCircle(A,a)
  \tkzDrawPoint(A)
  \tkzLabelPoints(A)
  \tkzLabelCircle[right=10pt](A,a)(-60)%
     {$\mathcal{C}_{(A~;~2)}$}
\end{tikzpicture}
```

2 Constructible Numbers

Although approximate by nature, the unmarked ruler and the compass are among the most precise instruments of classical geometry. Two arbitrary points O and I in the plane determine an axis endowed with a unit of length, namely the distance OI.
A ruler-and-compass construction starts from the two initial points O and I and proceeds step by step by constructing new points distinct from those already obtained. At each step, only the following operations are permitted:

– drawing the straight line determined by two previously constructed distinct points;

– drawing a circle with centre at a previously constructed point and radius equal to the distance between two previously constructed points.

The intersection points of such lines and circles define new points, called *constructible points*. By a slight abuse of language, one also speaks of constructible lines, circles, or sets of points.

2.1 Formal definition

Let $\mathcal{E}$ be a subset of the Euclidean plane, identified with $\mathbb{R}^2$. A point P with coordinates (x, y) is said to be *constructible in one step from* $\mathcal{E}$ if and only if either

$$\mathrm{P} \in \mathcal{E},$$

or P belongs to the intersection of two geometric objects chosen among:

– straight lines passing through two distinct points of $\mathcal{E}$;

– circles with centre at a point of $\mathcal{E}$ and radius equal to the distance between two points of $\mathcal{E}$.

We denote by $\mathcal{C}_1(\mathcal{E})$ the set of points constructible in one step from $\mathcal{E}$. If $\mathcal{E}$ is finite, then $\mathcal{C}_1(\mathcal{E})$ is also finite.

2.2 Points constructible in n steps

The notion of constructibility is defined inductively. The set of points constructible in n steps from $\mathcal{E}$ is given by

$$\mathcal{C}_1(\mathcal{E}) \quad \text{for } n = 1, \qquad \mathcal{C}_{n+1}(\mathcal{E}) = \mathcal{C}_1(\mathcal{C}_n(\mathcal{E})).$$

– More explicitly, let $\mathcal{E}$ be a finite set of points in the plane. Consider the collection $\mathcal{P}$ consisting of all straight lines determined by pairs of points of $\mathcal{E}$, together with all circles centred at a point of $\mathcal{E}$ and having as radius the distance between two points of $\mathcal{E}$. The *points constructed from* $\mathcal{E}$ *by ruler and compass* are the intersection points of the elements of $\mathcal{P}$.

– A point M in the plane is said to be *constructible* if there exists a finite sequence of points

$$\mathrm{M}_1, \mathrm{M}_2, \dots, \mathrm{M}_n = \mathrm{M}$$

such that, for every $i \le n$, the point M_i is constructible in one step from the set

$$\{\mathrm{O}, \mathrm{I}, \mathrm{M}_1, \dots, \mathrm{M}_{i-1}\}.$$

2.3 Authorized instruments and operations

Classical ruler-and-compass constructions rely on the following rules:

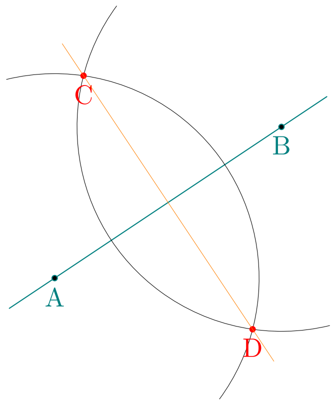
1. the ruler allows one to draw the straight line through any two given points;
2. the compass allows one to draw a circle with a given centre and a given radius;
3. new points arise as intersection points of two lines, of a line and a circle, or of two circles.

3 Some classic constructions

3.1 Dividing a Segment into Two Equal Parts

3.1.1 General method

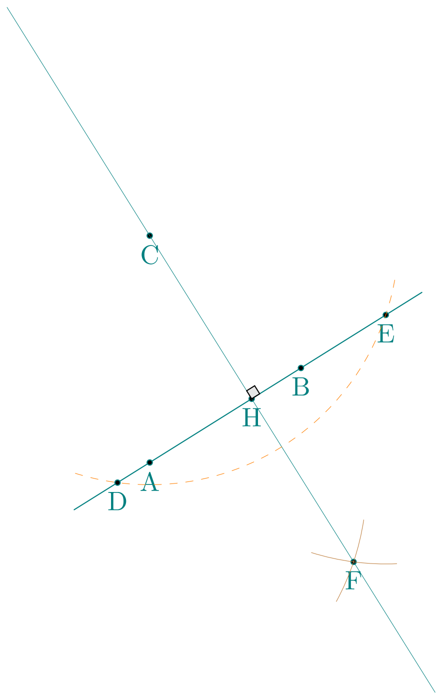
Draw the line (AB). Place one point of the compass at one end of the segment, at point A, and with the other point of the same compass, opened to a width greater than half the segment [AB], describe two arcs intersecting the line at points C and D. Then, keeping the same opening of the compass, place one point at the other end of the segment, at point B, and with the other point make it intersect the two arcs previously drawn at C and D. From the two intersection points—that is, from the points where the arcs intersect—draw the line (CD); it cuts the segment [AB] into two equal parts.
Remark: *Specific method - Simply draw arcs with centres* A *and* B *and radius AB.*




```
\directlua{
 init_elements()
 z.A = point(0, 0)
 z.B = point(3, 2)
 L.AB = line(z.A, z.B)
 L.med = L.AB:mediator()
 z.a, z.b = L.med:get()
 lenAB = L.AB.length
 angAB = math.deg(L.AB.slope)
 tkzRadius = 3 / 4 * lenAB
 C.A = circle(through(z.A, tkzRadius))
 C.B = circle(through(z.B, tkzRadius))
 z.C, z.D = intersection(C.A, C.B)
}
\begin{center}
\begin{tikzpicture}
  \tkzGetNodes
  \tkzDrawLine(A,B)
  \tkzDrawPoints(A,B)
  \tkzLabelPoints(A,B)
  \def\tkzRadius{\tkzUseLua{tkzRadius}}
  \begin{scope}[rotate = \tkzUseLua{angAB}]
    \tkzDrawArc[R](B,\tkzRadius)(120,240)
    \tkzDrawArc[R](A,\tkzRadius)(-60,60)
  \tkzDrawPoints[red](C,D)
  \tkzLabelPoints[red](C,D)
 \end{scope}
 \tkzDrawLine[orange,ultra thin](a,b)
\end{tikzpicture}
\end{center}
```

3.2 From a point, drop a perpendicular onto a line

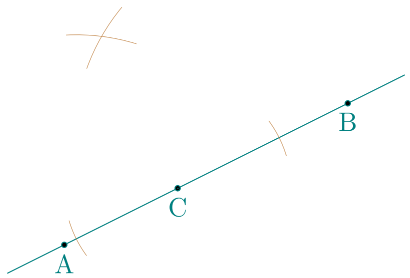
Place the compass point at C and draw an arc intersecting the line (AB) at the points D and E. With the same opening, draw two arcs centered at D and E, intersecting at a point F. Draw the line (CF); it intersects the line (AB) at a point H and is perpendicular to the line (AB). Thus, the perpendicular to (AB) passing through the point C has been constructed.




```
\directlua{
 init_elements()
 z.A = point(0, 0)
 z.B = point(4, 2.5)
 z.C = point(0, 6)
 L.AB = line(z.A, z.B)
 z.H = L.AB:projection(z.C)
 local R = tkz.length(z.C, z.H) + 1.5
 C.ED = circle(through(z.C, R))
 z.D, z.E = intersection(L.AB, C.ED)
 C.D = circle(through(z.D, R))
 C.E = circle(through(z.E, R))
 z.F = intersection(C.D, C.E, {known = z.C})
}

\begin{tikzpicture}[scale=.5]
  \tkzGetNodes
  \tkzDrawLine[add= .5 and .8](A,B)
  \tkzDrawPoints(A,...,F,H)
  \tkzLabelPoints(A,...,F,H)
  \tkzDrawLine[add = .7 and .4,new](C,F)
  \tkzMarkRightAngle[fill=gray!20](C,H,B)
  \tkzDrawArc[orange,dashed](C,D)(E)
  \tkzCompasss(D,F E,F)
\end{tikzpicture}
```

3.3 Given a line, erect a perpendicular to it at a given point

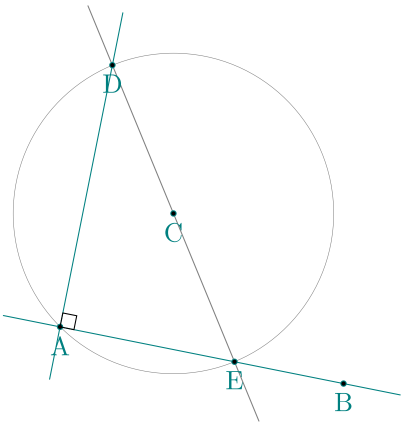
Let C be a given point. With centre C and any radius, describe a circle cutting the line (AB) at the points e and f. With centres e and f and the same radius, describe two circles intersecting at a point D. Join C to D.
Then the straight line (CD) is perpendicular to the straight line (AB) at the point C.




```
\directlua{
 init_elements()
 z.A = point(0, 0)
 z.B = point(5, 2.5)
 z.C = point(2, 1)
 L.AB = line(z.A, z.B)
 z.H = L.AB:projection(z.C)
}

\begin{center}
\begin{tikzpicture}[scale=.75]
  \tkzGetNodes
  \tkzDrawLine(A,B)
  \tkzDrawPoints(A,B,C)  \tkzLabelPoints(A,B,C)
  \tkzDrawLine[add = .5 and .1](C,H)
  \tkzShowLine[orthogonal=through C,size=2,gap=3](A,B)
\end{tikzpicture}
\end{center}
```

3.4 To erect a perpendicular to a given line at a point on the line

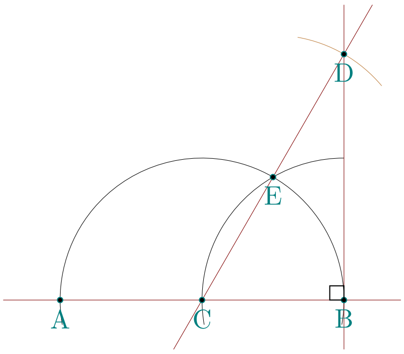
Let A be a given point on the straight line (AB). Take any point C not lying on (AB). With centre C and radius CA, describe a circle cutting the line (AB) at the point E. Join E to C and produce the straight line (EC) until it meets the circle again at D. Join A to D.
Then the straight line (AD) is perpendicular to the straight line (AB) at the point A.




```
\directlua{
 init_elements()
 z.A = point(0, 0)
 z.B = point(5, -1)
 z.C = point(2, 2)
 C.CA = circle(z.C, z.A)
 L.AB = line(z.A, z.B)
 z.E = intersection(L.AB, C.CA, {known = z.A})
 L.EC = line(z.E, z.C)
 z.D = intersection(L.EC, C.CA, {known = z.E})
}
\begin{center}
\begin{tikzpicture}[scale=.75]
  \tkzGetNodes
  \tkzDrawLine(A,B)
  \tkzDrawCircle(C,A)
  \tkzDrawLine[color=gray,add=1.4 and .4](C,D)
  \tkzDrawLine(A,D)
  \tkzDrawPoints(A,B,E,C,D)
  \tkzMarkRightAngle(B,A,D)
  \tkzLabelPoints(A,B,E,C,D)
\end{tikzpicture}
\end{center}
```

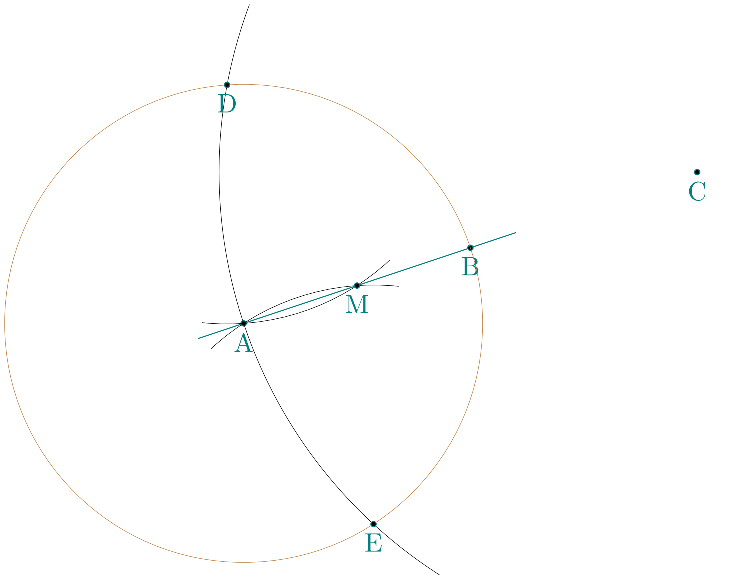
3.5 Another method for the same problem

Let the straight line (AB) be drawn. With centre B and any radius BC, and with centre B and C, describe two arcs intersecting at a point E. Join C to E and produce the straight line (CE) indefinitely. On this produced line, set off from E a length ED equal to CE. Join B to D.
Then the straight line (BD) is perpendicular to the straight line (AB) at the point B.




```
\directlua{
 init_elements()
 z.A = point(0, 0)
 z.B = point(5, 0)
 z.C = point(2.5, 0)
 z.E = z.C:rotation(math.pi / 3, z.B)
 z.D = z.E:symmetry(z.C)
}

\begin{center}
\begin{tikzpicture}[scale=.75]
  \tkzGetNodes
  \tkzSetUpLine[color=Maroon]
  \tkzDrawArc[angles](C,B)(0,180)
  \tkzDrawArc[angles,](B,C)(100,180)
  \tkzCompass[delta=20](E,D)
  \tkzDrawLines(A,B C,D B,D)
  \tkzDrawPoints(A,...,E)
  \tkzLabelPoints(A,B,C,D,E)
  \tkzMarkRightAngle(C,B,D)
\end{tikzpicture}
\end{center}
```

3.6 To find the midpoint of a given segment by means of the compass



```
\directlua{
 init_elements()
 z.A = point(0, 0)
 z.B = point(6, 2)
 z.C = z.B:symmetry(z.A)
 L.AB = line(z.A, z.B)
 C.AB = circle(z.A, z.B)
 C.CA = circle(z.C, z.A)
  z.D, z.E = intersection(C.AB, C.CA)
 C.DA = circle(z.D, z.A)
 C.EA = circle(z.E, z.A)
 z.M = intersection(C.DA, C.EA, {known = z.A})
}
\begin{center}
\begin{tikzpicture}[scale=.5]
  \tkzGetNodes
  \tkzDrawLine(A,B)
  \tkzDrawPoints(A,B,C)
  \tkzLabelPoints(A,B,C)
  \tkzDrawArc[delta=10](D,A)(M)
  \tkzDrawArc[delta=10](E,M)(A)
  \tkzDrawArc[delta=10](C,D)(E)
  \tkzDrawCircle[color=brown,line width=.2pt](A,B)
   \tkzDrawPoints(D,E,M)
   \tkzLabelPoints(D,E,M)
 \end{tikzpicture}
\end{center}
```

3.7 Trisection of a segment

Let A and B be two distinct points. The problem is to construct points that divide the segment [AB] into three equal parts.

3.7.1 Method 1

$$\mathrm{AK} = \frac{1}{3}\mathrm{AB}$$

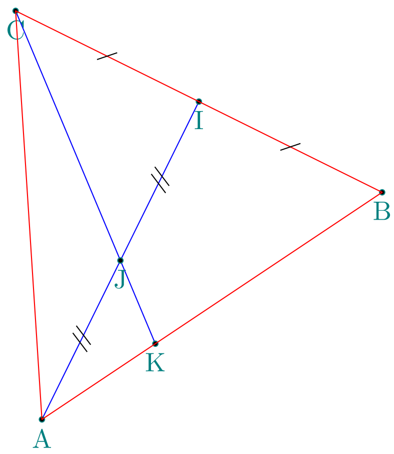
Construction.

1. Construct an equilateral triangle ABC on the segment [AB].
2. Let I be the midpoint of [BC].
3. Let J be the midpoint of [AI].
4. Draw the line (CJ).
5. Let K be the intersection point of (CJ) with the line (AB).

Justification. In an equilateral triangle, midpoint constructions and parallelism lead to similar triangles. The successive midpoint constructions imply

$$\mathrm{AK} = \frac{1}{3}\mathrm{AB}.$$

This construction uses only straightedge and compass operations and relies on symmetry and similarity.




```
\directlua{
 init_elements()
 z.A = point(0, 0)
 z.B = point(3, 2)
 L.AB = line(z.A, z.B)
 T.ABC = L.AB:equilateral()
 z.C = T.ABC.pc
 z.I = T.ABC.bc.mid
 z.J = tkz.midpoint(z.A, z.I)
 L.CJ = line(z.C, z.J)
 z.K = intersection(L.CJ, T.ABC.ab)
}
\begin{center}
\begin{tikzpicture}[scale=1.5]
  \tkzGetNodes
  \tkzDrawSegments[color=blue](C,K A,I)
  \tkzDrawPoints(A,B,C,J,I,K)
  \tkzLabelPoints(A,B,C,J,I,K)
  \tkzDrawPolygon[red](A,B,C)
  \tkzMarkSegments[mark=s|](C,I I,B)
  \tkzMarkSegments[mark=s||](A,J J,I)
\end{tikzpicture}
\end{center}
```

3.7.2 Method 2

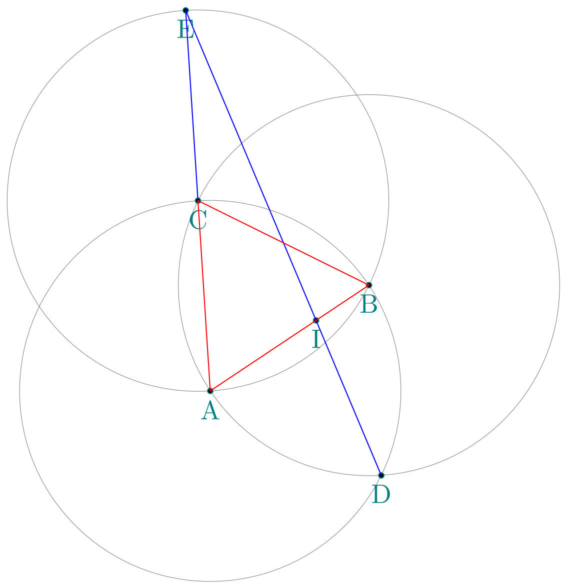
$$\mathrm{BI} = \frac{1}{3}\mathrm{AB}$$

Construction.

1. Draw the circles with centres A and B and radius AB.
2. Let C be one of their intersection points.
3. Draw the circle with centre C and radius CA.
4. Let E be the second intersection of this circle with the line (AC).
5. Draw the line (DE), where D is the second intersection of the first two circles.
6. Let I be the intersection point of (DE) with the line (AB).

Justification. The construction produces a configuration of equilateral triangles and intersecting chords. By similarity and symmetry arguments, the point I satisfies

$$\mathrm{BI} = \frac{1}{3}\mathrm{AB}.$$




```
\directlua{
 z.A = point(0, 0)
 z.B = point(3, 2)
 L.AB = line(z.A, z.B)
 C.AB =  circle(z.A, z.B)
 C.BA =  circle(z.B, z.A)
 z.C, z.D = intersection(C.AB, C.BA)
 z.E = intersection(line(z.A, z.C),
       circle(z.C, z.A), {known = z.A})
  z.I = intersection(line(z.D, z.E), L.AB)}

\begin{tikzpicture}[scale=.7]
\tkzGetNodes
 \tkzDrawCircle(C,A)
 \tkzDrawCircle(A,B)
 \tkzDrawCircle(B,A)
 \tkzDrawPoints(A,...,E,I)
 \tkzLabelPoints(A,...,E,I)
 \tkzDrawSegments[blue](D,E C,E)
 \tkzDrawPolygon[red](A,B,C)
\end{tikzpicture}
```

3.7.3 Method 3

$$\mathrm{AI} = \frac{1}{3}\mathrm{AB}$$

This method completes and complements Method 1.

Construction.

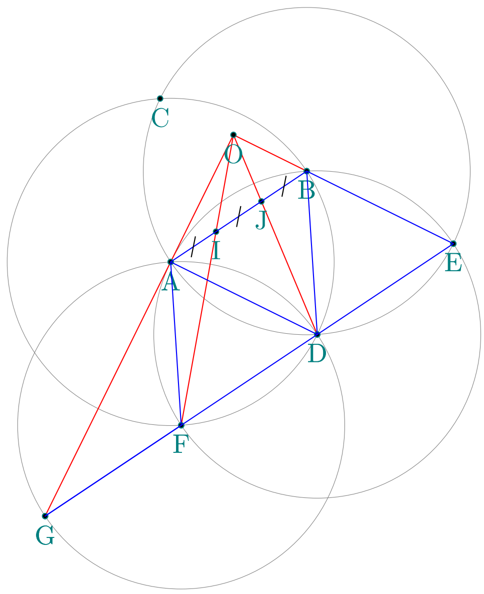
1. Construct the circles with centres A and B and radius AB.
2. From their intersection points, construct auxiliary circles and intersection points D, E, F and G.
3. Draw the lines (AG) and (BE), and let O be their intersection.
4. Draw the lines (OD) and (OF).
5. Let J and I be the intersection points of these lines with (AB).

Justification. The construction creates a system of similar triangles and parallel segments. The marked equal segments show that

$$\mathrm{AI} = \mathrm{IJ} = \mathrm{JB},$$

and therefore

$$\mathrm{AI} = \frac{1}{3}\mathrm{AB}.$$




```
\directlua{
 z.A = point(0, 0)
 z.B = point(3, 2)
 L.AB = line(z.A, z.B)
 C.AB =  circle(z.A, z.B)
 C.BA =  circle(z.B, z.A)
 z.C, z.D = intersection(C.AB, C.BA)
 C.DB = circle(z.D, z.B)
 z.E = intersection(C.DB, C.BA,{known= z.A})
 z.F = intersection(C.DB, C.AB,{known= z.B})
 L.EF = line(z.E, z.F)
 C.FA = circle(z.F, z.A)
 z.G = intersection(L.EF, C.FA, {known = z.D})
 z.O = intersection(line(z.A, z.G), line(z.B, z.E))
 z.J = intersection(line(z.O, z.D), L.AB)
 z.I = intersection(line(z.O, z.F), L.AB)
 }

  \begin{tikzpicture}[scale=.6]
   \tkzGetNodes
   \tkzDrawCircle(D,A)
   \tkzDrawCircle(A,B)
   \tkzDrawCircle(B,A)
   \tkzDrawCircle(F,A)
   \tkzDrawSegments[color=red](O,G O,B)
   \tkzDrawSegments[color=red](O,D O,F)
   \tkzDrawPoints(A,B,D,E,F,G,I,J,O)
   \tkzLabelPoints(A,B,D,E,F,G,I,J,O)
   \tkzDrawPoints(A,B,C,D,E,F,G)
   \tkzLabelPoints(A,B,C,D,E,F,G)
   \tkzDrawSegments[blue](A,B B,D A,D)
   \tkzDrawSegments[blue](A,F F,G E,G B,E)
   \tkzMarkSegments[mark=s|](A,I I,J J,B)
\end{tikzpicture}
```

Remark. The three constructions are valid straightedge-and-compass solutions. They illustrate different geometric principles:

- Method 1: midpoints and similarity,
- Method 2: circular symmetry and equilateral configurations,
- Method 3: symmetry and completion of the trisection.

3.8 Construction of the square root of a number

Square roots are therefore constructible.
Let $a > 0$. On a line, place three points $\mathrm{O}, \mathrm{I}, \mathrm{A}$ such that

$$\mathrm{OI} = 1 \qquad \text{and} \qquad \mathrm{OA} = a.$$

Then $\mathrm{IA} = a - 1$ and the segment $[\mathrm{OA}]$ has length a.

1. Construct the midpoint M of the segment $[\mathrm{OA}]$.

2. Draw the circle with center M passing through A (equivalently, the circle with diameter [OA]).
3. At the point I, erect the perpendicular to the line (OA); call this perpendicular line (IB).
4. Let B be the intersection of this perpendicular with the circle.

Claim. The length IB is $\sqrt{a}$.

Proof. Since B lies on the circle with diameter [OA], the angle $\angle$OBA is a right angle (Thales' theorem), hence the triangle OBA is right-angled at B. Because IB $\perp$ OA, the segment IB is the altitude from B to the hypotenuse OA of the right triangle OBA.
In a right triangle, the altitude to the hypotenuse satisfies the geometric mean relation

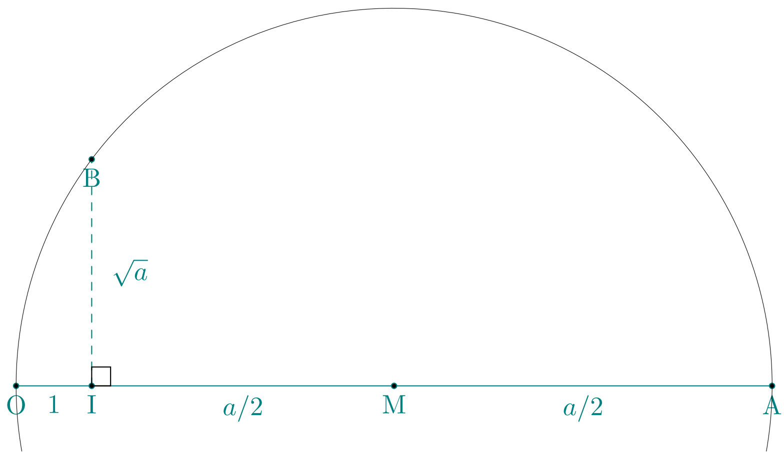
$$\mathrm{IB}^2 = \mathrm{OI} \cdot \mathrm{IA}.$$

Here OI $= 1$ and IA $= a$ (this is exactly the convention used in the figure: the point I fixes the unit, and A encodes the number a on the same ray), hence

$$\mathrm{IB}^2 = 1 \cdot a = a, \qquad \text{so} \qquad \mathrm{IB} = \sqrt{a}.$$

Therefore the segment [IB] represents the square root of a.

□



```
\directlua{
 init_elements()
 z.O = point(0, 0)
 z.I = point(1, 0)
 z.A = point(10, 0)
 L.OA = line(z.O, z.A)
 z.M = L.OA.mid
 z.H = z.I:north()
 C.MA = circle(z.M, z.A)
 L.IH = line(z.I, z.H)
 z.B = intersection(L.IH, C.MA)
}

\begin{tikzpicture}[scale=1]
  \tkzGetNodes
   \tkzDrawSegment(O,A)
   \tkzDrawSegment[style=dashed](I,B)
   \tkzDrawPoints(O,I,A,M,B)
   \tkzLabelPoints(O,I,A,M,B)
    \tkzDrawArc(M,A)(O)
    \tkzMarkRightAngle(A,I,B)
   \tkzLabelSegment[right=4pt](I,B){$\sqrt{a}$}
   \tkzLabelSegment[below](O,I){$1$}
   \tkzLabelSegment[below](I,M){$a/2$}
   \tkzLabelSegment[below](M,A){$a/2$}
\end{tikzpicture}
```

4 Triangles

4.1 Equilateral triangle

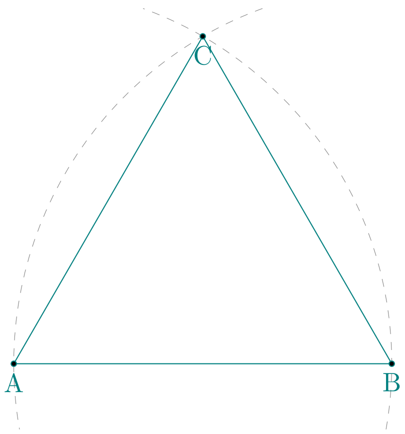
No particular difficulties:
Let A and B be two distinct points.
Draw the circle with center A and radius AB, and the circle with center B and radius AB. These two circles intersect at a point C.
By construction, we have

$$AC = AB \quad \text{and} \quad BC = AB.$$

It follows that

$$AB = BC = CA.$$

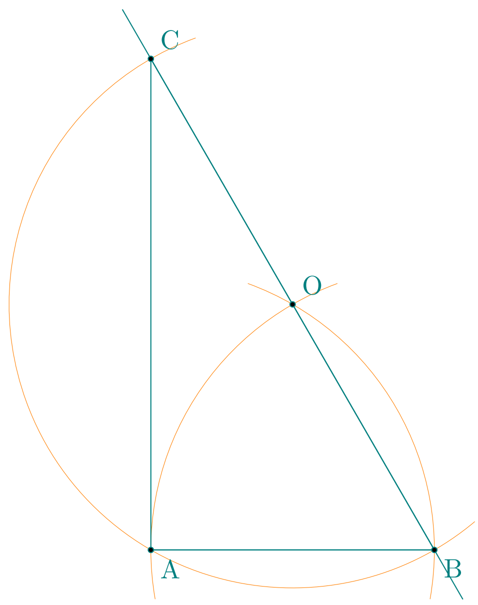
Therefore, the points A, B, and C form an equilateral triangle constructed on the segment [AB].




```
\directlua{
 init_elements()
 z.A = point(0, 0)
 z.B = point(5, 0)
 L.AB = line(z.A, z.B)
 C.AB = circle(z.A, z.B)
 C.BA = circle(z.B, z.A)
 z.C = intersection(C.AB, C.BA)
}

\begin{tikzpicture}
 \tkzGetNodes
 \tkzDrawPolygon(A,B,C)
 \tkzDrawArc[color=gray,%
            style=dashed,%
            delta=10](A,B)(C)
 \tkzDrawArc[color=gray,%
       style=dashed,delta=10](B,C)(A)
  \tkzDrawPoints(A,B,C)
  \tkzLabelPoints(A,B,C)
\end{tikzpicture}
```

4.2 School triangle

A school triangle is a right triangle with one angle equal to 60°. It may be constructed by drawing an equilateral triangle and erecting a perpendicular at one of its vertices. A more economical construction is obtained by reflecting one vertex of the equilateral triangle across another vertex.

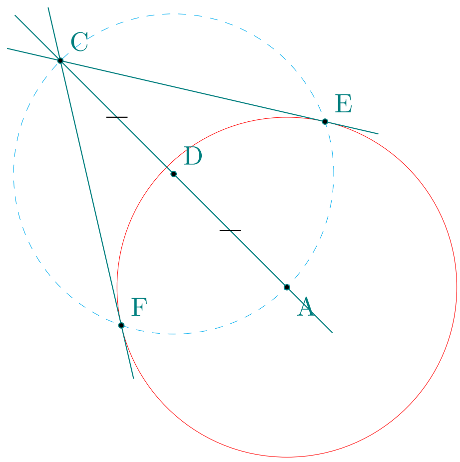



```
\directlua{
 init_elements()
 z.A = point(0, 0)
 z.B = point(5, 0)
 L.AB = line(z.A, z.B)
 C.AB =  circle(z.A, z.B)
 C.BA =  circle(z.B, z.A)
 z.O = intersection(C.AB, C.BA)
 z.C = intersection(line(z.B, z.O),
      circle(z.O, z.B), {known = z.B})
}

\begin{tikzpicture}[scale=.75]
  \tkzGetNodes
   \tkzDrawSegment(A,B)
  \tkzDrawArc[color=orange,delta=10](A,B)(O)
  \tkzDrawArc[color=orange,delta=10](B,O)(A)
  \tkzDrawArc[rotate,color=orange,delta=10](O,B)(-
180)
  \tkzDrawLine[add=.2 and 1.2](B,O)
  \tkzDrawPolygon(A,B,C)
  \tkzDrawPoints(A,B,C,O)
  \tkzLabelPoints[below right](A,B)
  \tkzLabelPoints[above right](O,C)
\end{tikzpicture}
```

5 Circles

5.1 From a pointv, to draw tangents to a given circle

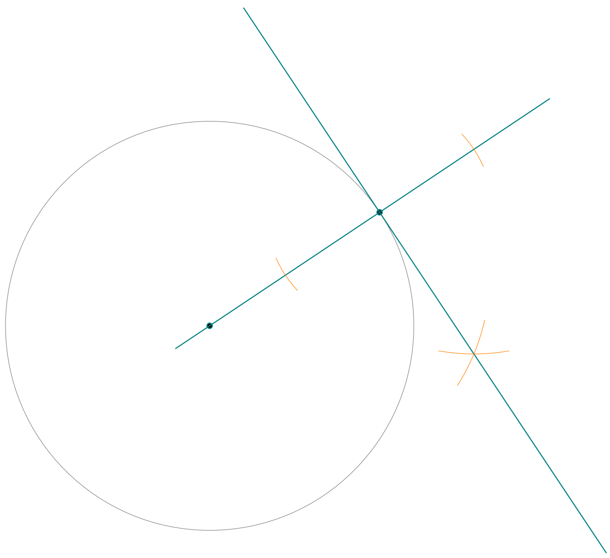
From the centre A of the given circle, draw the line (AC) and divide the segment [AC] into two equal parts. Let D be the midpoint of [AC]. With centre D and radius DC, describe a circle which intersects the given circle at points E and F. Draw the lines (CE) and (CF); these lines are tangents to the given circle, each touching it at a single point, namely E and F.
Indeed, each tangent is perpendicular to the corresponding radius, [AE] or [AF]; without this condition, the line would not be tangent to the given circle.




```
\directlua{
 init_elements()
 z.A = point(0, 0)
 z.B = point(3, 0)
 C.AB = circle(z.A, z.B)
 z.C = point(-4, 4)
 L.AC = line(z.A, z.C)
 z.D = L.AC.mid
 C.DC = circle(z.D, z.C)
 z.E, z.F = intersection(C.AB, C.DC)
}

\begin{tikzpicture}[scale=.75]
  \tkzGetNodes
  \tkzDrawCircle[red](A,B)
  \tkzDrawCircle[cyan,dashed](D,C)
  \tkzDrawLines(A,C C,E C,F)
  \tkzDrawPoints(A,C,D,E,F)
  \tkzLabelPoints[below right](A)
  \tkzLabelPoints[above right](C,D,E,F)
  \tkzMarkSegments[mark=s|](C,D D,A)
\end{tikzpicture}
```

5.2 From a point A on the circumference of a circle, to draw a tangent to the circle

From the centre C of the given circle, draw the segment [AC], which is a radius of the circle. At the endpoint A of this radius, erect a perpendicular, using the method of Figure 4; let this perpendicular be the line (BD). This line is tangent to the given circle and touches it at the single point A.



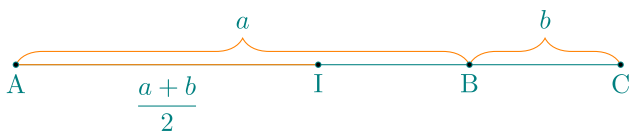
```
\directlua{
 init_elements()
 z.C = point(0, 0)
 z.A = point(3, 2)
 C.CA = circle(z.C, z.A)
 L.TA = C.CA:tangent_at(z.A)
 z.a, z.b = L.TA:get()
}

\begin{tikzpicture}[scale=.75]
  \tkzGetNodes
  \tkzDrawPoints(C,A)
  \tkzDrawLine[add = .2 and 1](C,A)
  \tkzDrawCircle(C,A)
  \tkzDrawLine[add = .5 and .1](a,b)
  \tkzShowLine[orthogonal=through A,%
     size=2,color=orange,gap=3](A,C)
\end{tikzpicture}
```

6 Construction of Means

6.1 Arithmetic Mean

On a line (Δ), place two segments $[AB]$ and $[BC]$ of respective lengths a and b. Let I be the midpoint of the segment $[AC]$. Then

$$IA = IC = \frac{a+b}{2}.$$




```
\directlua{
  z.A = point(0, 0)
  z.B = point(6, 0)
  z.C = point(8, 0)
  z.Ap = point(0, -1)
  z.Ip = point(4, -1)
  z.I = tkz.midpoint(z.A, z.C)
}

\begin{center}
\begin{tikzpicture}
  \tkzGetNodes
  \tkzInit[xmin = -1,ymin = -3,xmax = 9,ymax = 1]
  \tkzClip
  \tkzDrawSegment[decoration={brace,amplitude=10pt},
     decorate,color=orange](A,B)
  \tkzDrawSegment[decoration={brace,amplitude=10pt},
     decorate,color=orange](B,C)
   \tkzDrawSegment(A,C)
  \tkzDrawSegment[decoration={brace,amplitude=20pt},
      postaction={decorate},color=orange](I,A)
  \tkzDrawPoints(A,B,C,I)\tkzLabelPoints(A,B,C,I)
  \tkzLabelSegment[above=10pt](A,B){$a$}
  \tkzLabelSegment[above=10pt](B,C){$b$}
  \tkzLabelSegment[above](A',I'){$\dfrac{a+b}{2}$}
\end{tikzpicture}
\end{center}
```

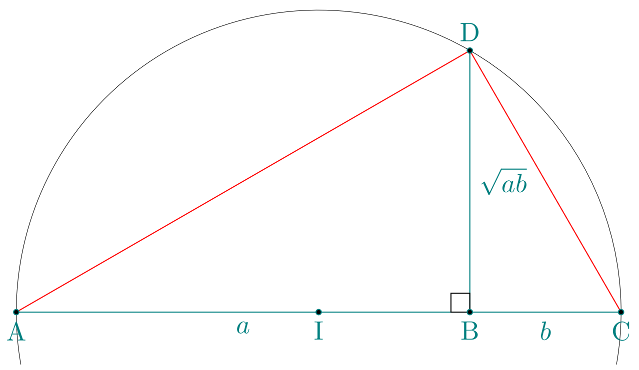
6.2 Geometric Mean

On a line (Δ), place two segments $[AB]$ and $[BC]$ of respective lengths a and b. Let D be a point on the circle with diameter $[AC]$ such that the line (BD) is perpendicular to $[AC]$ at B. The triangle ADC is right-angled at D, which implies that the right triangles ADB and BDC are similar. Hence

$$\frac{\mathrm{BD}}{\mathrm{AB}} = \frac{\mathrm{BC}}{\mathrm{BD}}, \quad \text{that is,} \quad \mathrm{BD}^2 = \mathrm{AB} \times \mathrm{BC} = a \times b.$$

Finally,

$$\mathrm{BD} = \sqrt{ab}.$$



```
\directlua{
  z.A = point(0, 0)
  z.B = point(6, 0)
  z.C = point(8, 0)
  z.Ap = point(0, -1)
  z.Ip = point(4, -1)
  z.I = tkz.midpoint(z.A, z.C)
  L.AB = line(z.A, z.B)
  L.BH = L.AB:orthogonal_from(z.B)
  C.IC = circle(z.I, z.C)
  z.D =  intersection(L.BH, C.IC)
}

\begin{center}
\begin{tikzpicture}
  \tkzGetNodes
  \tkzInit[xmin = -1,ymin = -1,
           xmax = 11,ymax = 6]
  \tkzClip
  \tkzDrawArc(I,C)(A)
  \tkzDrawSegments(A,C B,D)
  \tkzDrawSegments[red](A,D C,D)
  \tkzMarkRightAngle(A,B,D)
  \tkzDrawPoints(A,B,C,D,I)
  \tkzLabelSegment[right](B,D){$\sqrt{ab}$}
  \tkzLabelPoints(A,B,C,I)
  \tkzLabelPoints[above](D)
  \tkzLabelSegment[below](A,B){$a$}
  \tkzLabelSegment[below](B,C){$b$}
\end{tikzpicture}
\end{center}
```

6.3 Harmonic Mean

If

$$\frac{2}{c} = \frac{1}{a} + \frac{1}{b},$$

then c is called the *harmonic mean* of a and b.
This relation can also be written as

$$c = \frac{2ab}{a+b}.$$

Given two numbers a and b, let $\mathcal{A}$, $\mathcal{G}$, and $\mathcal{H}$ denote respectively their arithmetic, geometric, and harmonic means. We now show that

$$\mathcal{G}^2 = \mathcal{H} \times \mathcal{A}.$$

- Let a and b be two numbers such that $\mathrm{OA} = a$ and $\mathrm{AB} = b$. Let I be the centre of the circle $\mathcal{C}$ with diameter [OB]. Let [IK] be a radius of this circle perpendicular to (OB). Then

$$\mathrm{IK} = \mathcal{A}.$$

- The line (AG) is perpendicular to (OB) at A, and G is a point on the circle $\mathcal{C}$. The triangle OGB is right-angled, hence

$$\mathrm{AG}^2 = \mathrm{OA} \times \mathrm{OB}.$$

Therefore

$$\mathrm{AG} = \mathcal{G} = \sqrt{ab}.$$

The triangles GAH and IAG are similar right triangles (see the construction of the geometric mean). Hence,

$$\frac{\mathrm{GH}}{\mathrm{AG}} = \frac{\mathrm{AG}}{\mathrm{IH}}, \qquad \text{that is,} \qquad \frac{\mathrm{AG}}{\mathcal{G}} = \frac{\mathcal{G}}{\mathcal{A}}.$$

Thus,

$$\mathcal{G}^2 = \mathcal{A} \times \mathrm{AG}, \qquad \text{and} \qquad \mathrm{AG} = \frac{\mathcal{G}^2}{\mathcal{A}}.$$

We may therefore conclude that

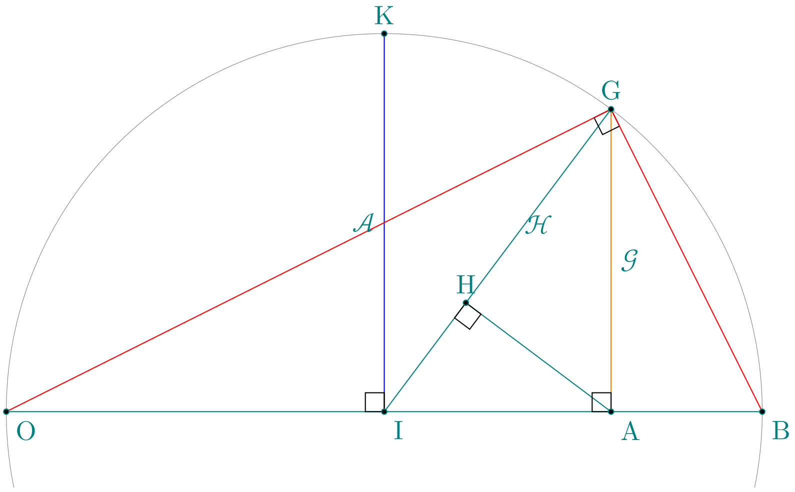
$$\mathrm{AG} = \mathcal{H}.$$

Indeed,

$$\frac{1}{\mathcal{H}} = \frac{\mathcal{A}}{\mathcal{G}^2} = \frac{a+b}{2ab},$$

and finally,

$$\frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) = \frac{1}{\mathcal{H}}.$$



```
\directlua{
z.O = point(0, 0)
z.A = point(8, 0)
z.B = point(10, 0)
z.I = point(5, 0)
z.K = point(5, 5)
L.AB = line(z.A, z.B)
L.AG = L.AB:orthogonal_from(z.A)
C.IK = circle(z.I, z.K)
z.G = intersection(L.AG, C.IK)
L.IG = line(z.I, z.G)
z.H = L.IG:projection(z.A)
}

\begin{center}
\begin{tikzpicture}
  \tkzGetNodes
  \tkzInit[xmin = -1,ymin = -1,xmax = 11,ymax = 6]
  \tkzClip
  \tkzDrawSegments(I,H A,H H,G O,B)
  \tkzDrawSegment[color = blue](I,K)
  \tkzDrawSegment[color = orange](A,G)
  \tkzDrawCircle(I,K)
  \tkzMarkRightAngles(O,I,K O,A,G A,H,I O,G,B)
  \tkzDrawSegments[red](O,G G,B)
  \tkzDrawPoints(O,A,B,G,K,H,I)
  \tkzLabelSegment[left](I,K){$\mathcal{A}$}
  \tkzLabelSegment[right](A,G){$\mathcal{G}$}
  \tkzLabelSegment(H,G){$\mathcal{H}$}
  \tkzLabelPoints[below right](O,A,B,I)
  \tkzLabelPoints[above](G,H,K)
\end{tikzpicture}
\end{center}
```

6.4 Constructing two segments from their sum and their geometric mean

Knowing the geometric mean of two segments amounts to knowing the product of their lengths. Let s and g denote respectively the sum and the geometric mean of two positive lengths a and b.
In the figure below, the segment $[\mathrm{AC}]$ has length

$$\mathrm{AC} = a + b = s,$$

and the segment $[\mathrm{AM}]$ has length

$$\mathrm{AM} = g = \sqrt{ab}.$$

Knowing the geometric mean $g = \sqrt{ab}$ is equivalent to knowing the product

$$g^2 = ab.$$

To construct the segment of length g, draw:

- a circle with diameter $[\mathrm{AC}]$, where $\mathrm{AC} = s$;
- a second circle with diameter $[\mathrm{OB}]$, where $\mathrm{OB} = 1 + g^2$, and with $\mathrm{OA} = 1$.

The tangent at A to the first circle intersects the second circle at a point M such that

$$\mathrm{AM} = g = \sqrt{ab}.$$

Through M, draw a line parallel to (AC). Under suitable conditions, this line intersects the circle with diameter [AC] at a point P. Let S be the foot of the perpendicular from P onto the line (AC). In the right triangle APC, the altitude [PS] satisfies

$$\frac{\mathrm{AS}}{\mathrm{PS}} = \frac{\mathrm{PS}}{\mathrm{SC}},$$

hence

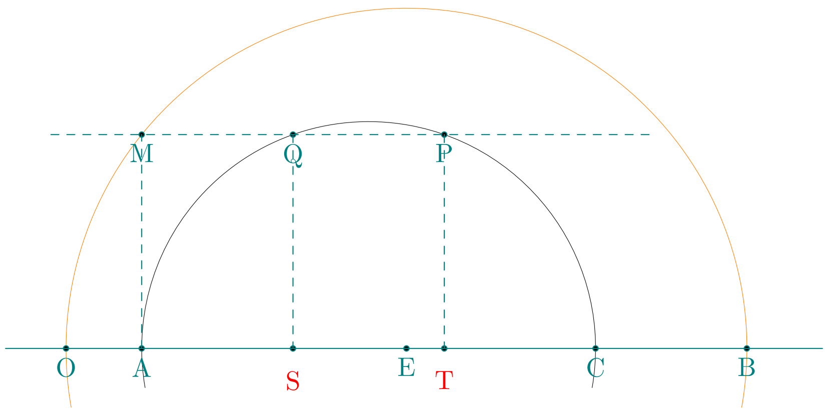
$$\mathrm{AS} \times \mathrm{SC} = \mathrm{PS}^2 = \mathrm{AM}^2 = g^2 = ab.$$

Moreover,

$$\mathrm{AS} + \mathrm{SC} = \mathrm{AC} = a + b.$$

Therefore, the segments [AS] and [SC] (with SC = AT in the figure) are the two solutions of the quadratic equation

$$x^2 - (a+b)x + ab = 0.$$



```
\directlua{
  local mg = 8
  local sum = 6
  z.O = point(-1, 0)
  z.A = point(0, 0)
  z.B = point(8, 0)
  L.OB = line(z.O, z.B)
  z.E = L.OB.mid
  z.C = point(sum, 0)
  L.AC = line(z.A, z.C)
  z.I = L.AC.mid
  C.IA = circle(z.I, z.A)
  local sqmg = math.sqrt(mg)
  z.M = (z.C - z.A):orthogonal(sqmg):at(z.A)
  L.pp = L.AC:parallel_from(z.M)
  z.P, z.Q = intersection(L.pp, C.IA)
  z.T, z.S = L.AC:projection(z.P, z.Q)
}
\begin{center}
\begin{tikzpicture}
  \tkzGetNodes
  \tkzDrawArc(I,C)(A)
   \tkzDrawArc[orange](E,B)(O)
   \tkzDrawPoints(O,A,B,C,E,M,P,Q,S,T)
   \tkzLabelPoints(O,A,B,C,E,M,P,Q)
   \tkzDrawLine[add =.3 and .5](A,C)
   \tkzDrawLine[add =.3 and .7,dashed](M,P)
   \tkzDrawSegments[style=dashed](A,M P,T Q,S)
   \tkzLabelPoints[below=5pt,color=red](S,T)
\end{tikzpicture}
\end{center}
```

It is not necessary to know the algebraic solution of a quadratic equation in order to determine these two lengths. Two purely geometric approaches are possible.

- **First method.** Assume $a \geq b$. Then

  $$(a-b)^2 = a^2 - 2ab + b^2 = (a+b)^2 - 4ab,$$

  which gives

  $$(a-b)^2 = s^2 - 4g^2, \qquad a - b = \sqrt{s^2 - 4g^2}.$$

  Remark. The condition $s^2 - 4g^2 \geq 0$ is equivalent to

  $$\frac{(a+b)^2}{4} \geq ab,$$

  that is,

  $$\frac{a+b}{2} \geq \sqrt{ab}.$$

  This expresses the classical inequality stating that the arithmetic mean is greater than or equal to the geometric mean. Under this condition, the line through M parallel to (AC) indeed intersects the circle with diameter $[\mathrm{AC}]$.

- **Second method.** Let m be the half-sum of a and b, so that $m = s/2$. Assuming again $a \geq b$, there exists a real number $\alpha \geq 0$ such that

  $$a = m + \alpha, \qquad b = m - \alpha.$$

Using the knowledge of the geometric mean, we write

$$g^2 = ab = (m + \alpha)(m - \alpha) = m^2 - \alpha^2.$$

Hence

$$\alpha^2 = m^2 - g^2, \qquad \alpha = \sqrt{m^2 - g^2} = \sqrt{\frac{s^2}{4} - g^2}.$$

6.5 Constructing two segments from their difference and their geometric mean

Let d and g be two given positive numbers. We wish to construct two segments of lengths x and y such that

$$y - x = d \quad \text{and} \quad xy = g^2.$$

In the figure below, the segments $[\mathrm{MP}]$ and $[\mathrm{MQ}]$ provide the required solution. Indeed,

$$\mathrm{MQ} - \mathrm{MP} = \mathrm{PQ} = d \quad \text{and} \quad \mathrm{MP} \times \mathrm{MQ} = \mathrm{MA}^2 = g^2.$$

Geometric construction.

1. Draw a segment $[\mathrm{AB}]$ and construct its midpoint O.
2. Draw the circle with center O and radius OA; thus $[\mathrm{AB}]$ is a diameter of the circle.
3. Choose a point M outside the circle and draw the line (MO).
4. Let P and Q be the intersection points of the line (MO) with the circle.
5. Project the points P and Q orthogonally onto the line (AB); denote the projections by P' and Q'.

Justification.
Since P and Q lie on the circle with diameter $[\mathrm{AB}]$, the angles $\angle\mathrm{APB}$ and $\angle\mathrm{AQB}$ are right angles (Thales' theorem). Consequently, the triangles MAP and MAQ are right triangles with altitude MA.
In a right triangle, the altitude to the hypotenuse satisfies the geometric mean relation. Applied here, this yields

$$\mathrm{MP} \times \mathrm{MQ} = \mathrm{MA}^2.$$

If the segment MA is chosen so that $\mathrm{MA} = g$, then

$$\mathrm{MP} \times \mathrm{MQ} = g^2.$$

Moreover, since P and Q are symmetric with respect to O, their projections P' and Q' onto (AB) satisfy

$$\mathrm{PQ} = \mathrm{Q}'\mathrm{P}' = d,$$

so that

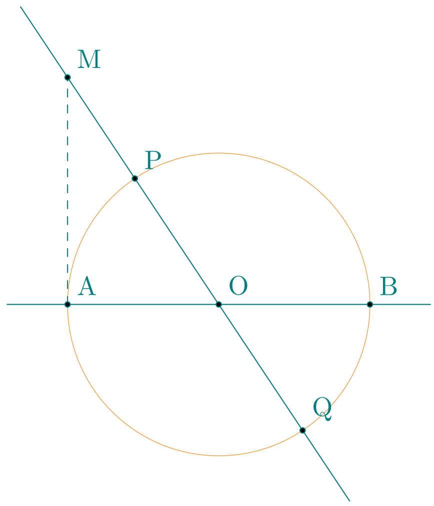
$$\mathrm{MQ} - \mathrm{MP} = d.$$

Thus, the two segments $[\mathrm{MP}]$ and $[\mathrm{MQ}]$ have the prescribed difference d and geometric mean g.

Remark. This construction provides a purely geometric solution to the system

$$\begin{cases} y - x = d, \\ xy = g^2, \end{cases}$$

without any recourse to the algebraic solution of a quadratic equation.

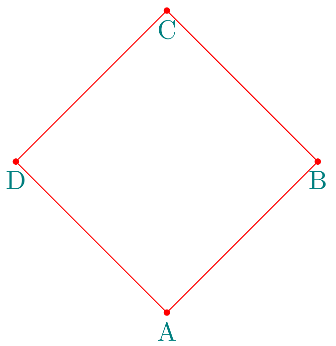



```
\directlua{
 z.A = point(0, 0)
 z.B = point(4, 0)
 z.M = point(0, 3)
 L.AB = line(z.A, z.B)
 z.O = L.AB.mid
 C.OA = circle(z.O, z.A)
 L.MO = line(z.M, z.O)
 z.P, z.Q = intersection(L.MO, C.OA)
 z.Pp, z.Qp = L.AB:projection(z.P, z.Q)
}
\begin{center}
\begin{tikzpicture}
  \tkzGetNodes
  \tkzDrawCircle[color=orange](O,B)
  \tkzDrawLines(A,B M,Q)
  \tkzDrawSegment[style=dashed](A,M)
  \tkzDrawPoints(O,A,B,P,Q,M)
  \tkzLabelPoints[above right](O,A,B,P,Q,M)
\end{tikzpicture}
\end{center}
```

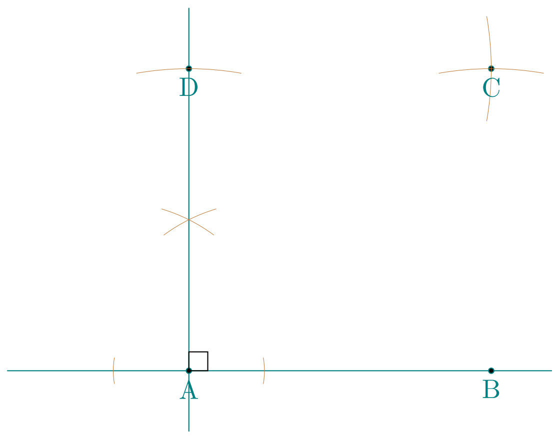
7 Construction of Squares

7.1 Method 1 and tkz-euclide Tool

Draw a segment [AB] of arbitrary length, which will be the side of the square. At point B, erect a perpendicular to (AB) using the method described previously, and mark on this perpendicular a segment [BC] such that BC = AB.
From point A, with a compass opening equal to AB, draw an arc. From point C, with the same compass opening, draw a second arc intersecting the first one at a point H.
Finally, draw the segments [AH] and [CH]. The quadrilateral ABCH is a square.




```
\directlua{
  init_elements()
  z.A = point(0, 0)
  z.B = point(2, 2)
  L.AB = line(z.A, z.B)
  S.ABCD = L.AB:square()
  _,_,z.C,z.D = S.ABCD:get()
}

\begin{tikzpicture}
  \tkzGetNodes
  \tkzDrawPolygon[color=red](A,B,C,D)
  \tkzDrawPoints[color=red](A,B,C,D)
  \tkzLabelPoints(A,B,C,D)
\end{tikzpicture}
```

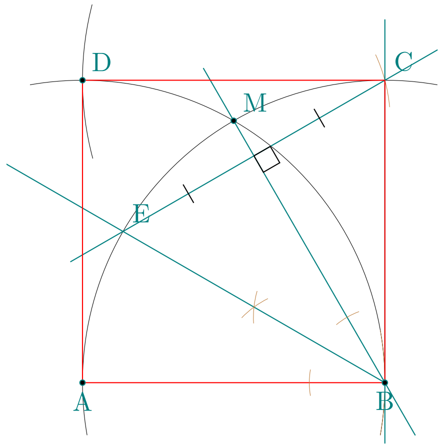



```
\directlua{
  init_elements()
  z.A = point(0, 0)
  z.B = point(4, 0)
  L.AB = line(z.A, z.B)
  L.AD = L.AB:orthogonal_from(z.A)
  z.D = L.AD.pb
  dAB  = L.AB.length
  z.C = L.AB:parallel_from(z.D).pb
}

\begin{center}
\begin{tikzpicture}
  \tkzGetNodes
  \tkzDrawLine[add= .6 and .2](A,B)
  \tkzDrawLine(A,D)
  \tkzDrawPoints(A,B,C,D)
  \tkzMarkRightAngle(B,A,D)
  \tkzCompasss(D,C B,C A,D)
  \tkzShowLine[orthogonal= through A,gap=2](A,B)
  \tkzLabelPoints(A,B,C,D)
\end{tikzpicture}
\end{center}
```

7.2 Method 2: Compass Construction

Draw the line (AB). With center A and radius AB, draw an indefinite arc greater than a quarter of a circle. With center B and the same compass opening AB, draw a similar arc intersecting the first one at a point C.

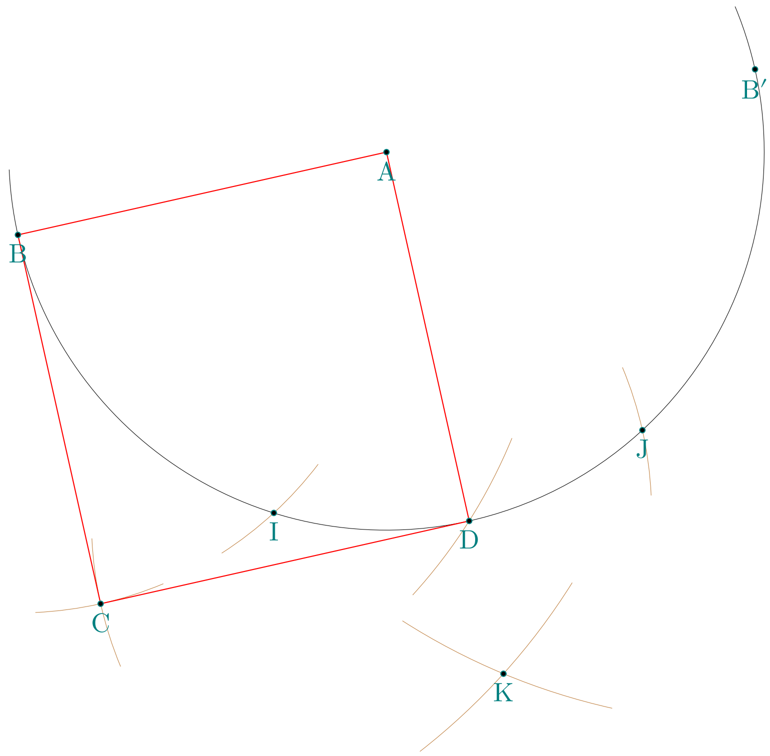
Divide the arc AC into two equal parts; let D be the point of division. Using the compass, transfer the arc CD from point C to a point F, and transfer the same arc CD from point C to a point E.
Finally, draw the lines (AE), (EF), and (BF). Together with the line (AB), these lines form a square.




```
\directlua{
  init_elements()
  z.A = point(0, 0)
  z.B = point(4, 0)
  L.AB = line(z.A, z.B)
  T.ABM = L.AB:equilateral()
  z.M = T.ABM.pc
  L.bis = T.ABM:bisector(z.B)
  z.b = L.bis.pb
  L.Bb = line(z.B, z.b)
  C.BA = circle(z.B, z.A)
  z.E, z.F = intersection(L.Bb, C.BA)
  L.BM = line(z.B, z.M)
  z.C = L.BM:reflection(z.E)
  L.CE = line(z.C, z.E)
  z.H = intersection(L.BM, L.CE)
  C.AB = circle(z.A, z.B)
  C.CB = circle(z.C, z.B)
  z.D, z.K = intersection(C.CB, C.AB)
}
\begin{tikzpicture}
  \tkzGetNodes
  \tkzInit[xmin=-1,xmax=5,ymin=-1,ymax=5]
  \tkzClip
  \tkzDrawArc[rotate,delta=10](A,B)(90)
  \tkzDrawArc[angles,delta=10](B,A)(90,180)
  \tkzDrawLine[add =0 and 5](B,b)
  \tkzDrawLine(B,M)
  \tkzCompasss(M,C A,B)
  \tkzDrawArc[angles,delta=10](C,B)(175,185)
  \tkzDrawLines(B,C E,C)
  \tkzDrawPolygon[color=red](A,B,C,D)
  \tkzMarkSegments[mark=|](E,H H,C)
  \tkzShowLine[bisector](M,B,A)
  \tkzLabelPoints(A,B)
  \tkzLabelPoints[above right](M,E,C,D)
  \tkzMarkRightAngle(B,H,C)
  \tkzDrawPoints(A,B,M,D)
\end{tikzpicture}
```

7.3 Method using only a compass

This method is particularly interesting because the compass keeps the same opening throughout the entire construction.


B′
A
B
J
I
D
C
K

```
\directlua{
  init_elements()
  z.A = point(0, 0)
  C.A5 = circle(through(z.A, 5))
  z.B = C.A5:random()
  z.Bp = z.A:symmetry(z.B)
  C.B5 = circle(through(z.B, 5))
  z.Ip, z.I = intersection(C.A5, C.B5)
  if tkz.is_direct(z.A, z.B, z.I) then
  else
  z.I, z.Ip = z.Ip, z.I
  end
  C.I5 = circle(through(z.I, 5))
  z.J = intersection(C.A5, C.I5, {known = z.B})
 C.Bp5 = circle(through(z.Bp, 5))
 C.BJ = circle(z.B, z.J)
 C.BpI = circle(z.Bp, z.I)
 z.K = intersection(C.BJ, C.BpI, {near = z.I})
 C.AB = circle(z.A, z.B)
 C.BD = circle(through(z.B, tkz.length(z.A, z.K)))
 z.D = intersection(C.AB, C.BD, {near = z.I})
 C.BA = circle(z.B, z.A)
 C.DA = circle(z.D, z.A)
 z.C = intersection(C.BA, C.DA, {known = z.A})
 }
\begin{center}
  \begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawArc(A,B)(B')
   \tkzCompasss(B,I I,J B,K B',K B,D B,C D,C)
   \tkzDrawPolygon[color=red](A,B,C,D)
   \tkzDrawPoints(A,B,B',I,J,K,D,C)
   \tkzLabelPoints(A,B,B',I,J,K,D,C)
\end{tikzpicture}
 \end{center}
```

8 Construction of the pentagon

Euclid constructs a regular pentagon (both equilateral and equiangular) inscribed in a circle. His basic building block is the *golden triangle*: an isosceles triangle whose base angles are twice the vertex angle (so the vertex angle is one fifth of a straight angle). Thus,

$$\frac{180^\circ}{5} = 36^\circ.$$

8.1 Construction of the golden triangle

In the accompanying figure, I is the midpoint of $[AC]$, $AC = AB$, $IB = ID$, and $AD = AE = BF$. Euclid proves that triangle ABF is a golden triangle using rather lengthy geometric arguments (see Wikipedia).
Nowadays, the proof is simpler. If we set $AC = 1$, we obtain

$$IB = \frac{\sqrt{5}}{2},$$

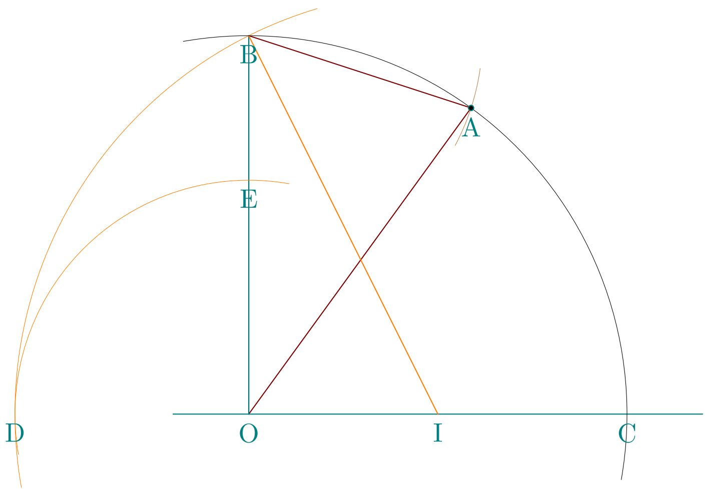
and

$$AD = AE = BF = \frac{\sqrt{5}-1}{2} = \frac{1}{\Phi},$$

by the Pythagorean theorem, where

$$\Phi = \frac{\sqrt{5}+1}{2}$$

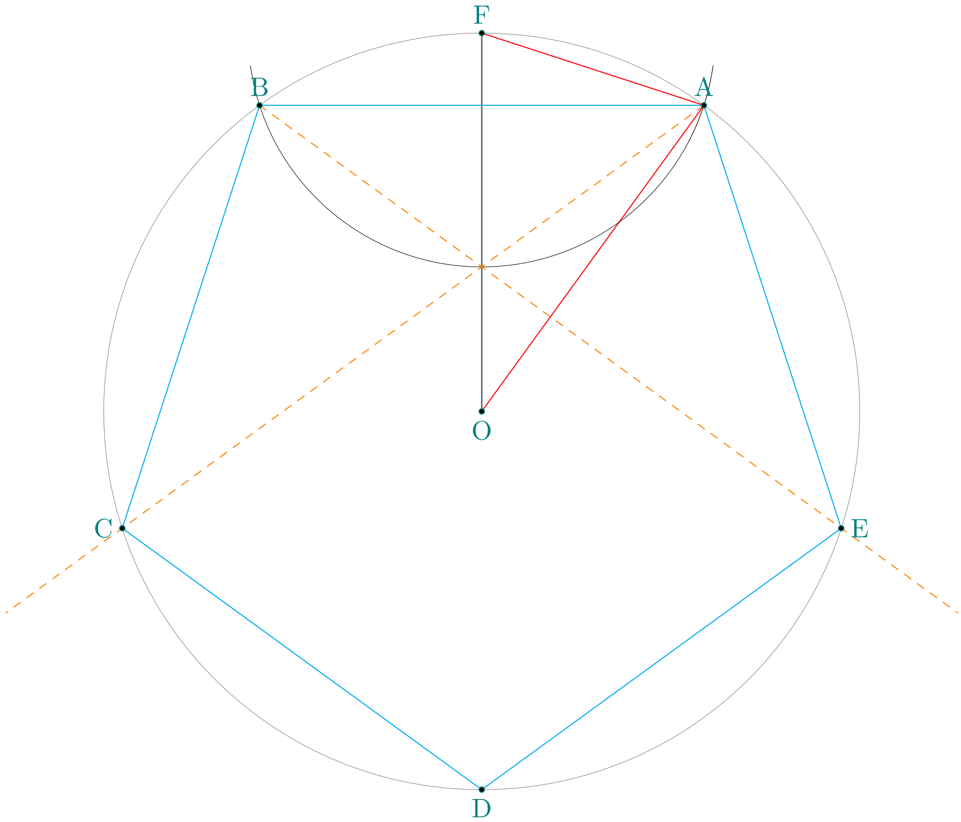
is the golden ratio.
The side lengths of triangle ABF are therefore 1, 1, and $\frac{1}{\Phi}$. Hence, ABF is indeed a golden triangle.



```
\directlua{
 z.O = point(0, 0)
 z.B = point(0, 5)
 z.C = point(5, 0)
 L.OC = line(z.O, z.C)
 L.OB = line(z.O, z.B)
 z.I = L.OC.mid
 z.D = L.OC:report(-tkz.length(z.I, z.B),z.I)
 z.E = L.OB:report(tkz.length(z.O, z.D),z.O)
 C.OC = circle(z.O, z.C)
 C.BA = circle(through(z.B, tkz.length(z.O, z.E)))
 z.A =  intersection(C.OC, C.BA, {near = z.B} )
}

\begin{center}
\begin{tikzpicture}
 \tkzGetNodes
 \tkzDrawPolygon[color=Maroon](O,B,A)
 \tkzDrawLine(O,C)
 \tkzDrawSegment(O,B)
 \tkzDrawSegment[color=orange](I,B)
 \tkzCompass(B,A)
 \tkzDrawArc[color=orange](I,B)(D)
 \tkzDrawArc[color=orange](O,E)(D)
 \tkzDrawArc(O,C)(B)
 \tkzDrawPoint(A)
 \tkzLabelPoints(O,A,B,C,D,I,E)
\end{tikzpicture}
\end{center}
```

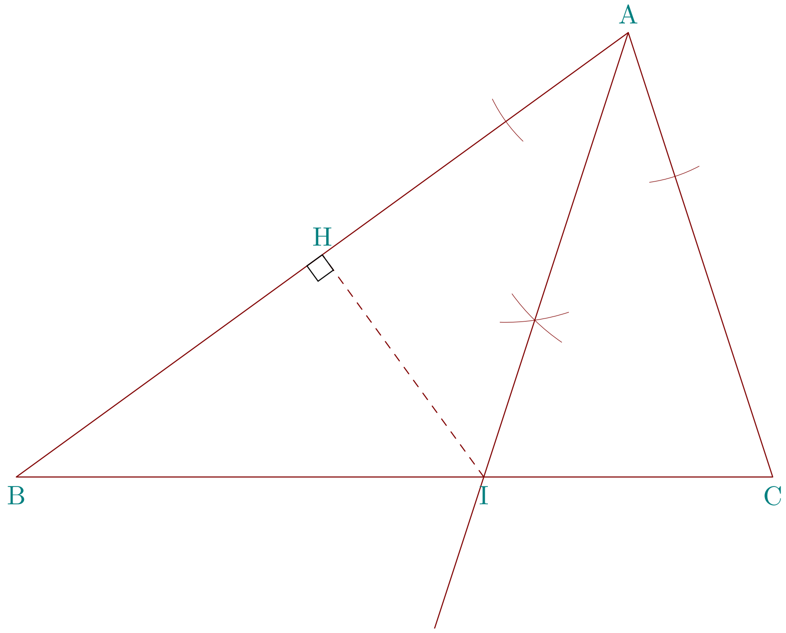
8.2 Construction of the pentagon using a golden triangle

Euclid proves that it is possible to construct a pentagon inside a circle. Starting from the golden triangle OAF, one constructs the golden triangle ABD by drawing the circular arc with centre F and radius FA. By taking the angle bisectors of $\widehat{A}$ and $\widehat{B}$ and extending them until they meet the circle, the two missing vertices C and E are obtained.


F
B
A
O
C
E
D

```
\directlua{
 z.O = point(0, 0)
 z.F = point(0, 5)
 z.c = point(5, 0)
 L.Oc = line(z.O, z.c)
 L.OF = line(z.O, z.F)
 z.I = L.Oc.mid
 z.d = L.Oc:report(-tkz.length(z.I, z.F),z.I)
 z.e = L.OF:report(tkz.length(z.O, z.d),z.O)
 C.Oc = circle(z.O, z.c)
 C.FA = circle(through(z.F, tkz.length(z.O, z.e)))
 z.A, z.B =  intersection(C.Oc, C.FA, {near = z.F} )
 z.D = z.O:symmetry(z.F)
 T.ABD = triangle(z.A, z.B, z.D)
 L.A = T.ABD:bisector(z.A)
 L.B = T.ABD:bisector(z.B)
 z.C = intersection(L.A, C.Oc, {known=z.A})
 z.E = intersection(L.B, C.Oc, {known=z.B})
}
\begin{center}
\begin{tikzpicture}
 \tkzGetNodes
 \tkzDrawPolygon[color=red](O,F,A)
 \tkzDrawSegment(O,F)
 \tkzCompass(F,A)
 \tkzDrawCircle(O,A)
 \tkzDrawArc(F,B)(A)
 \tkzDrawPolygon[color=cyan](A,...,E)
 \tkzDrawSegments[add = 0 and .2,orange, dashed](A,C B,E)
 \tkzDrawPoints(A,B,F,C,D,O,E)
 \tkzLabelPoints(O,D)
 \tkzLabelPoints[above](A,B,F)
 \tkzLabelPoints[left](C)
 \tkzLabelPoints[right](E)
\end{tikzpicture}
\end{center}
```

8.3 Computation of $\cos\left(\frac{\pi}{5}\right)$




```
  \directlua{
  z.B = point(0, 0)
  z.C = point(10, 0)
  z.A = z.B:rotation(math.pi/5, z.C)
  T.ABC = triangle(z.A, z.B, z.C)
  L.Aa = T.ABC:bisector(z.A)
  z.I = intersection(L.Aa, T.ABC.bc)
  z.H = T.ABC.ab:projection(z.I)
  }
\begin{center}
\begin{tikzpicture}
  \tkzGetNodes
  \tkzInit[xmin=-1,xmax=11,ymin=-1,ymax=6]
  \tkzClip[space=1]
  \tkzShowLine[bisector,color=Maroon,size=2,gap=4](B,A,C)
  \tkzDrawLine[color=Maroon,add= 0 and 6](A,I)
  \tkzDrawPolygon[color=Maroon](A,B,C)
  \tkzDrawSegment[color=Maroon,style=dashed](I,H)
  \tkzMarkRightAngle(B,H,I)
  \tkzLabelPoints(B,C,I)
  \tkzLabelPoints[above](A,H)
\end{tikzpicture}
\end{center}
```

Let ABC be a golden triangle with base AC, such that

* $BA = BC$,
* $\widehat{ABC} = \dfrac{\pi}{5}$,

* [AI) is the bisector of $\widehat{BAC}$,
* $AC = 1$.

We then show that:

* $AC = AI = BI = 1$,
* $BA = 2BH = 2\cos\left(\frac{\pi}{5}\right)$,
* $IC = 2\cos\left(\frac{\pi}{5}\right) - 1$,
* $\frac{AC}{AB} = \frac{IC}{AC}$, hence

$$IC = \frac{1}{2\cos\left(\frac{\pi}{5}\right)}.$$

The last two equalities yield

$$2\cos\left(\frac{\pi}{5}\right)\left(2\cos\left(\frac{\pi}{5}\right) - 1\right) = 1,$$

that is,

$$4\cos^2\left(\frac{\pi}{5}\right) - 2\cos\left(\frac{\pi}{5}\right) - 1 = 0.$$

Letting $x = \cos\left(\frac{\pi}{5}\right)$, the positive solution of the equation

$$4x^2 - 2x - 1 = 0$$

is

$$\cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4} = \frac{\Phi}{2}.$$

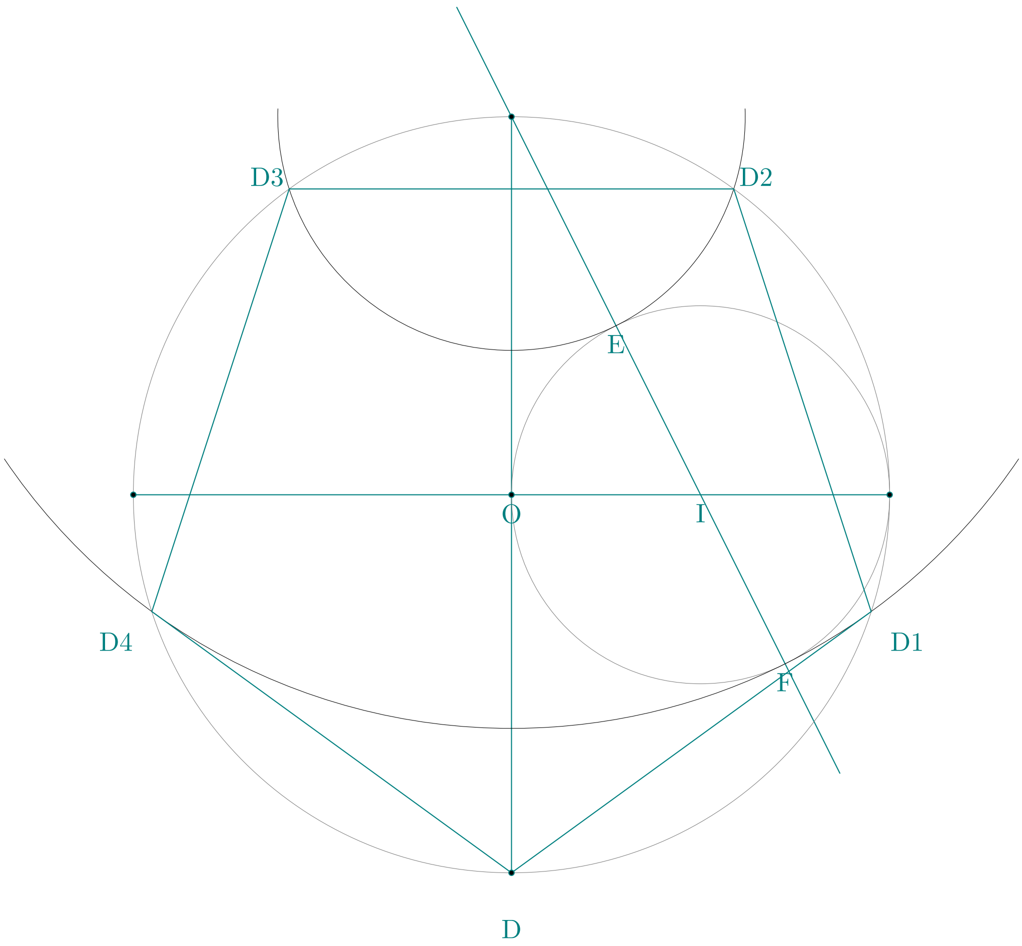
8.4 Pentagon inscribed in a circle

Euclid's construction can be greatly simplified by keeping the same principle: constructing golden or silver triangles.

1. Draw a circle Γ with centre O and radius R (arbitrary unit).
2. Draw two perpendicular diameters.

– Their intersections with Γ define the points A, B, C, D.
– Point A is diametrically opposite to C.
– Point B is diametrically opposite to D.
– Draw a circle Γ' with diameter [OA] (radius $R' = R/2$) and centre I. The circle Γ' passes through O and A.
– Draw a line (d) through B and I.
– The line (d) intersects the circle Γ' at points E and F (with E closer to B).
– Draw two (arcs of) circles Γ_1 and Γ_2 with centre B and radii BE and BF, respectively.
– The circles Γ_1 and Γ_2 intersect Γ at four points D_1, D_2, D_3, D_4.

– The points D, D_1, D_2, D_3, D_4 form a regular pentagon. Indeed, one checks that BOD_2 is a golden triangle and BOD_1 a silver triangle (their bases have the appropriate lengths, while their sides are equal to R).



```
\directlua{
 z.O = point(0, 0)
 z.A = point(5, 0)
 z.B = point(0, 5)
 z.C = point(-5, 0)
 z.D = point(0, -5)
 L.AO = line(z.A, z.O)
 z.I = L.AO.mid
 L.IB = line(z.I, z.B)
 C.OC = circle(z.O, z.C)
 z.E, z.F = intersection(L.IB, circle(z.I, z.A))
 z.D2, z.D3 = intersection(C.OC, circle(z.B, z.E))
 z.D4, z.D1 = intersection(C.OC, circle(z.B, z.F))
}
\begin{center}
\begin{tikzpicture}[scale=1]
  \tkzGetNodes
  \tkzDrawSegments(B,D C,A)
  \tkzDrawLine(B,F)
  \tkzDrawCircles(O,A I,O)
  \tkzDrawArc[delta=20](B,D3)(D2)
  \tkzDrawArc[delta=20](B,D4)(D1)
  \tkzDrawPolygon(D,D1,D2,D3,D4)
  \tkzDrawPoints(O,A,B,C,D)
  \tkzLabelPoints(E,F,O,I)
  \tkzAutoLabelPoints[center=O,dist=.1](D,D1,D2,D4,D3)
\end{tikzpicture}
\end{center}
```

9 Geometric construction of the rosette

9.1 Construction of a regular hexagon

Let O and A be two points in the plane. Draw the circle with centre O and passing through A.
Keeping the same compass opening, equal to OA, construct successively equilateral triangles on the circle: the circle with centre A intersects the given circle at a point p_1; the circle with centre p_1 intersects the given circle at a point p_2; and so on, until six points

$$p_0 = \mathrm{A},\ p_1,\ p_2,\ p_3,\ p_4,\ p_5$$

have been obtained on the circle.
The six points are such that

$$p_i p_{i+1} = \mathrm{OA} \qquad (i \bmod 6),$$

and therefore they form the vertices of a regular hexagon inscribed in the circle with centre O.

9.2 Construction of the rosette

Let (O, A) be a circle with centre O and radius OA.

1. Choose an arbitrary point p_0 on the circle (O, A).
2. With centre p_0 and radius $p_0\mathrm{O} = \mathrm{OA}$, draw a circle. This circle intersects (O, A) at two points. Choose one of them and call it p_1; the other one is denoted by p_5.
3. With centre p_1 and the same radius OA, draw a new circle. It intersects (O, A) at two points: one is p_0, the other is called p_2.
4. Repeat this construction successively:
   - the circle with centre p_2 intersects (O, A) at p_1 and p_3,
   - the circle with centre p_3 intersects (O, A) at p_2 and p_4,
   - the circle with centre p_4 intersects (O, A) at p_3 and p_5.
5. The six points $p_0, p_1, \ldots, p_5$ lie on the circle (O, A) and satisfy

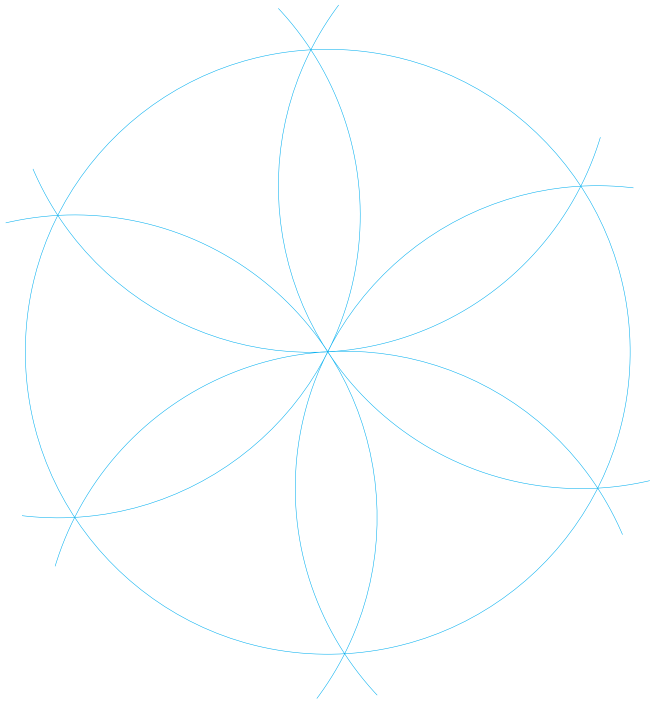
   $$p_i p_{i+1} = \mathrm{OA} \qquad (i \bmod 6),$$

   hence they form a regular hexagon inscribed in the circle.
6. For each vertex p_i of the hexagon, draw the circular arc with centre p_i and radius OA, joining the two neighbouring vertices p_{i-1} and p_{i+1} (indices taken modulo 6).

The six arcs thus obtained intersect at the centre O and form a symmetric six-petal rosette inside the circle (O, A).
Given two points in the plane. With the first point as centre and the second as point on the circumference, draw a circle. On the radius, construct an equilateral triangle. Repeating the same construction with the centre and the new point successively, a regular hexagon is obtained.

```
init_elements()
z.O = point(0, 0)
z.A = point(5, 0)
C.OA = circle(z.O, z.A)
z.p0 = C.OA:random()
```

```
local function rot(P, O, a)
  local x = P.re - O.re
  local y = P.im - O.im
  local ca = math.cos(a)
  local sa = math.sin(a)
  return point(O.re + ca*x - sa*y, O.im + sa*x + ca*y)
end

for i = 1, 5 do
  z["p"..i] = rot(z.p0, z.O, i*math.pi/3)
end
```



```
\directlua{dofile("tkz-geom-lua/rosace.lua")}
\begin{center}
\begin{tikzpicture}[scale=.8]
  \tkzGetNodes
  \tkzDrawCircle[color=cyan](O,A)
  \foreach \i/\j/\k in {0/1/5,
                      1/2/0,
                      2/3/1,
                      3/4/2,
                      4/5/3,
                      5/0/4}{%
    \tkzDrawArc[color=cyan](p\i,p\j)(p\k)}
  \end{tikzpicture}
\end{center}
```

10 A world map

This is a world map:

```
init_elements()
z.O = point(0, 0)
z.A = point(9, 0)
z.B = point(0, 9)
z.C = point(-9, 0)
z.D = point(0, -9)
local BD = line(z.B, z.D)
local AC = line(z.A, z.C)
PA.center1  = path:new()
PA.through1 = path:new()
PA.center2  = path:new()
PA.through2 = path:new()
PA.center3  = path:new()
PA.through3 = path:new()
PA.center4  = path:new()
PA.through4 = path:new()
local R1 = 9
for i = 1, 8 do
  local P  = point(R1*math.cos(math.rad(10*i)), R1*math.sin(math.rad(10*i)))
  local MP = point(0, i)
  local Lmp = line(MP, P)
  local M = tkz.midpoint(MP, P)
  local Lmed = Lmp:ortho_from(M)
  local Cc = intersection(BD, Lmed)
  PA.center1:add_point(Cc)
  PA.through1:add_point(P)
end
for i = 1, 8 do
  local P  = point(R1*math.cos(math.rad(-10*i)), R1*math.sin(math.rad(-10*i)))
  local MP = point(0, -i)
  local Lmp = line(MP, P)
  local M = tkz.midpoint(MP, P)
  local Lmed = Lmp:ortho_from(M)
  local Cc = intersection(BD, Lmed)
  PA.center2:add_point(Cc)
  PA.through2:add_point(P)
end
for i = 1, 8 do
  local NP = point(i, 0)
  local Lbn = line(z.B, NP)
  local M = tkz.midpoint(z.B, NP)
  local Lmed = Lbn:ortho_from(M)
  local Cc = intersection(AC, Lmed)
  PA.center3:add_point(Cc)
  PA.through3:add_point(NP)
end
for i = 1, 8 do
  local NP = point(-i, 0)
  local Lbn  = line(z.B, NP)
  local M = tkz.midpoint(z.B, NP)
  local Lmed = Lbn:ortho_from(M)
```

```
  local Cc = intersection(AC, Lmed)
  PA.center4:add_point(Cc)
  PA.through4:add_point(NP)
end
```